NASA TECHNICAL REPORT

NASA TR R-176

c.1

NASA TR R-176

A DISCUSSION OF HALPHEN'S METHOD FOR SECULAR PERTURBATIONS AND ITS APPLICATION TO THE DETERMINATION OF LONG RANGE EFFECTS IN THE MOTIONS OF CELESTIAL BODIES. PART 1.

by Peter Musen

Goddard Space Flight Center
Greenbelt, Maryland

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1963

A DISCUSSION OF HALPHEN'S METHOD FOR SECULAR PERTURBATIONS AND ITS APPLICATION TO THE DETERMINATION OF LONG RANGE EFFECTS IN THE MOTIONS OF CELESTIAL BODIES.

PART 1

By Peter Musen

Goddard Space Flight Center
Greenbelt, Maryland

Part II will be issued as NASA TR R-194

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce, Washington, D. C. 20230 -- Price $1.50

A DISCUSSION OF HALPHEN'S METHOD FOR SECULAR PERTURBATIONS AND ITS APPLICATION TO THE DETERMINATION OF LONG RANGE EFFECTS IN THE MOTIONS OF CELESTIAL BODIES. PART 1.

by
Peter Musen
Goddard Space Flight Center

SUMMARY

The long range effects caused by the moon and the sun are of primary importance in establishing the stability of highly eccentric satellite orbits. At present no complete analytical theory exists which can treat such orbits. It is shown here that Halphen's method of treating secular planetary effects can, by means of step-by-step integration, also be used to determine long range lunar effects in the motions of artificial satellites. Halphen's method permits the numerical integration of long range lunar effects over an interval of a few tens of years. The long range solar effects can be treated by averaging the disturbing function over the orbit of the satellite. Halphen's method is applicable to the determination of long range ("secular") effects in the motion of minor planets over the interval of hundreds of thousands of years. We assume that no sharp commensurability between mean motions of the disturbed and disturbing bodies does exist. A complete theory of Halphen's method is presented in modern symbols. Goursat transformations and a summability process are applied to speed the convergence of series which appear in the theory.

CONTENTS

A DISCUSSION OF HALPHEN'S METHOD FOR SECULAR PERTURBATIONS AND ITS APPLICATION TO THE DETERMINATION OF LONG RANGE EFFECTS IN THE MOTIONS OF CELESTIAL BODIES. PART 1.

by
Peter Musen
Goddard Space Flight Center

INTRODUCTION

In treating the problem of the orbital stability of celestial bodies, the long range effects are of primary importance; but no complete analytical theory considering these effects exists at present for large values of the eccentricity, inclination, and semimajor axis. To obtain information about the changes of its orbit over a long time interval, and hence information about the lifetime of, say, an artificial satellite, we have to resort to numerical integration.

Methods based on the use of an unaveraged disturbing function, such as those of Cowell or Encke, contain both the short and long period terms; and in the case of artificial satellites, they require that the interval of integration be much less than the satellite's period, thus creating a large accumulation of round-off errors. The main long range effects in the elements are produced by the long range terms in the disturbing function and by their cross-actions. The short period terms can also produce long range effects through their mutual cross-actions in higher order approximations, but such effects are very small (Reference 1) and can be neglected over a very long time interval. For these reasons, and to diminish the accumulation of round-off errors, it is necessary at the very beginning to remove the short period terms from the disturbing function or from the components of the disturbing force. The problem thus formulated does not differ from the problem of determining the secular perturbation of planets and comets by means of numerical integration using the Gaussian method (Reference 2). With the advent of modern electronic equipment, such a solution of the problem has become possible.

The use of Halphen's form of the Gaussian theory (Reference 3) was suggested by the author as a practical method for determining the long range effects through a step-by-step integration (Reference 4). Previously Halphen's method was not in use, *probably because of several numerical errors which appear in the original publication. They were all corrected by Goriachev (Reference 5),* whose name should be associated with the method as well; and in its present form the method should properly be called the Halphen-Goriachev method. Some parts of Halphen's original exposition can easily be recognized from the modern standpoint as an application of the calculus of dyadics (matrices) in a hidden form. In the present exposition we shall resort to vectors and matrices. The reason for this is not merely the wish to modernize the notations, but because the application of vectors and matrices removes all the ambiguities and difficulties connected with determining direction cosines when scalars are used. The latter problems are, on some occasions, sources of errors in Halphen's original presentation.

In Goriachev's work, all the formulas given in the final collection are correct; however, there are some misprints in the theoretical exposition. They are corrected here. The author has suggested (Reference 4) the use of the Goursat transformation (Reference 6) and of the E-summability process to speed the convergence of hypergeometric series which appear in the Halphen-Goriachev method and to facilitate the numerical computation.

A group working in celestial mechanics at Goddard Space Flight Center has applied Halphen's method of secular perturbations to the motions of planets, comets, and artificial satellites. The Halphen-Goriachev method was carefully compared with some other existing methods before it was recommended for large scale use. At present, no exposition of Halphen's method exists in English. In undertaking this exposition, the author was also motivated by the wish to present an interesting and important theory to the community of English-speaking astronomers.

SECULAR DISTURBING FUNCTION

The following notation will be used to describe the motion of the disturbed body:

$\mathbf{r}$ the position vector with respect to the central body,

$\mathbf{r}^0$ the unit vector in the direction of $\mathbf{r}$,

$\mathbf{P}$ the unit vector directed from the central body toward the osculating perigee,

$\mathbf{R}$ the unit vector, normal to the osculating orbit plane, in the direction of angular momentum,

$\mathbf{Q} = \mathbf{R} \times \mathbf{P}$,

e the osculating eccentricity,

a the semimajor axis,

$b = a\sqrt{1-e^2}$, the semiminor axis,

v the true anomaly,

E the eccentric anomaly,

g the mean anomaly.

Primed notations will be used to describe the motion of the disturbing body. The position vector of the disturbing body relative to the disturbed body will be designated by ρ, where $\rho = \mathbf{r}' - \mathbf{r}$.

Let m' be the mass of the disturbing body, f be the gravitational constant, and m be the mass of the disturbed body. The mass of the central body will be designated by M.

If the disturbing force

$$\mathbf{F} = fm'\left(\frac{\boldsymbol{\rho}}{\rho^3} - \frac{\mathbf{r}'}{r'^3}\right) \tag{1}$$

is developed into a double Fourier series with arguments g and g', then the constant term in the development is the secular disturbing force $[\mathbf{F}]$ and we have

$$[\mathbf{F}] = \frac{fm'}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\boldsymbol{\rho}}{\rho^3} - \frac{\mathbf{r}'}{r'^3}\right) dg\, dg' \, . \tag{2a}$$

Thus $[\mathbf{F}]$ is deduced from Equation 1 by applying a double process of averaging over the orbit of the disturbing body and over the orbit of the disturbed body.

Writing the "area integral" for the disturbing body in the form

$$dg' = \frac{r'^2\, dv'}{a'\, b'} \, , \tag{2b}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathbf{r}'}{r'^3}\, dg' = \frac{1}{2\pi a' b'}\int_0^{2\pi}\mathbf{r}^{0\prime}\, dv' = 0 \, .$$

Thus the indirect part $\left(-\mathbf{r}'/r'^3\right)$ of the disturbing force does not produce any secular effects and Equation 2a takes the form

$$[\mathbf{F}] = \frac{fm'}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\boldsymbol{\rho}}{\rho^3}\, dg\, dg' \, . \tag{3}$$

Let us first apply to $\mathbf{F}$ the process of averaging over the orbit of the disturbing body. This averaged force will be designated by $\mathbf{F}_0$:

$$\mathbf{F}_0 = \frac{fm'}{2\pi}\int_0^{2\pi}\frac{\boldsymbol{\rho}}{\rho^3}\, dg' \, . \tag{4}$$

In the process of determining $\mathbf{F}_0$ the position of the disturbing body is imagined to describe the complete osculating ellipse. However, we are interested neither in short period terms nor in knowing at what moment of time the disturbing body will occupy a particular position in its ellipse. This process of averaging is evidently a purely geometrical one.

The geometrical locus of vectors $\boldsymbol{\rho}$ is an elliptical cone with its apex in the disturbed body. Taking Equation 2b into account, we can also write

$$\mathbf{F}_0 = \frac{f m'}{2\pi a' b'} \int_0^{2\pi} \frac{\boldsymbol{\rho}}{\rho^3} r'^2 \, dv' \, . \tag{5}$$

If we consider two neighboring position vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho} + d\boldsymbol{\rho}$ with respect to the disturbed body m, then $r'^2 dv'/2$ represents the area of the elementary sector with the apex in the central body (Figure 1). Taking Equation 5 into account and setting

$$d\mu = \frac{m' r'^2 dv'}{2\pi a' b'} \, ,$$

we deduce

$$\mathbf{F}_0 = f \oint \frac{\boldsymbol{\rho}}{\rho^3} d\mu \, . \tag{6}$$

This integral is taken along the ellipse of the disturbing body in the direction of the motion. Equation 6 represents the Gaussian result: $\mathbf{F}_0$ *is equal to the attraction of an elliptic ring over which the mass is distributed proportionally to the area of the sector described by the radius vector* $\mathbf{r}'$.

Let $\boldsymbol{\rho}_0$ be the position vector of the central body relative to the disturbed body. Evidently

$$\boldsymbol{\rho}_0 = -\mathbf{r} \, .$$

Also let

$$h = \boldsymbol{\rho}_0 \cdot \mathbf{R}'$$

be the projection of $\boldsymbol{\rho}_0$ on $\mathbf{R}'$. We have

$$\boldsymbol{\rho} = \boldsymbol{\rho}_0 + \mathbf{r}' \, ,$$

$$d\mathbf{r}' = d\boldsymbol{\rho} \, ,$$

and

$$\boldsymbol{\rho}_0 \cdot \boldsymbol{\rho} \times d\boldsymbol{\rho} = \boldsymbol{\rho}_0 \cdot \left(\boldsymbol{\rho}_0 \times d\mathbf{r}' + \mathbf{r}' \times d\mathbf{r}'\right)$$

$$= \boldsymbol{\rho}_0 \cdot \mathbf{R}' r'^2 dv' = h r'^2 dv'$$

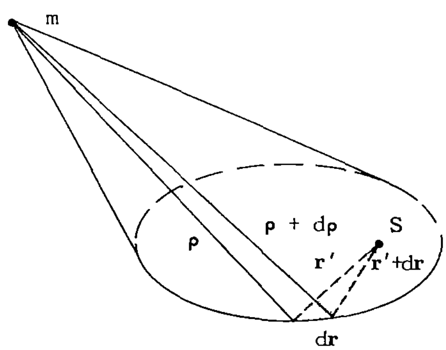

Figure 1—Area of the elementary sector.

or

$$r'^2 \, dv' = \frac{\boldsymbol{\rho}_0 \cdot \boldsymbol{\rho} \times d\boldsymbol{\rho}}{h} \, .$$

As a result of the last equation, Equation 5 becomes

$$\mathbf{F}_0 = \frac{fm'}{2\pi a' b' h} \oint \frac{\boldsymbol{\rho}\boldsymbol{\rho} \times d\boldsymbol{\rho}}{\rho^3} \cdot \boldsymbol{\rho}_0 \, . \tag{7}$$

Equation 7 introduces a dyadic

$$\Phi = \frac{1}{2} \oint \frac{\boldsymbol{\rho}\boldsymbol{\rho} \times d\boldsymbol{\rho}}{\rho^3} \tag{8}$$

closely associated with the problem of determining secular perturbations.

Designating the unit vector in the direction of $\boldsymbol{\rho}$ by $\boldsymbol{\rho}^0$ and substituting $\boldsymbol{\rho} = \rho \boldsymbol{\rho}^0$ into Equation 8, we can reduce Φ to the simpler form

$$\Phi = \frac{1}{2} \oint \boldsymbol{\rho}^0 \boldsymbol{\rho}^0 \times d\boldsymbol{\rho}^0 \, . \tag{9}$$

The peculiar characteristics of the dyadic Φ are that its first scalar invariant and its vector are both equal to zero.

We have

$$\Phi_s = \frac{1}{2} \oint \boldsymbol{\rho}^0 \cdot \boldsymbol{\rho}^0 \times d\boldsymbol{\rho}^0 = 0 \tag{10}$$

and

$$\Phi_\times = \frac{1}{2} \oint \boldsymbol{\rho}^0 \times \left(\boldsymbol{\rho}^0 \times d\boldsymbol{\rho}^0 \right) \, .$$

Developing the double cross-product in the form

$$\Phi_\times = \frac{1}{2} \oint \left(\boldsymbol{\rho}^0 \boldsymbol{\rho}^0 \cdot d\boldsymbol{\rho}^0 - \boldsymbol{\rho}^0 \cdot \boldsymbol{\rho}^0 \, d\boldsymbol{\rho}^0 \right)$$

and taking

$$\boldsymbol{\rho}^0 \cdot d\boldsymbol{\rho}^0 = 0 \, ,$$

$$\boldsymbol{\rho}^0 \cdot \boldsymbol{\rho}^0 = 1$$

into account, we deduce

$$\Phi_x = -\frac{1}{2}\oint d\rho^0 = 0 \cdot \tag{11}$$

The condition that the vector of a dyadic equal zero is necessary and sufficient for the symmetry of the dyadic. Consequently, Φ is symmetrical and Equation 7 can be written in the form

$$\mathbf{F}_0 = \frac{fm' \, \Phi \cdot \rho_0}{\pi a' b' h} = \frac{fm' \, \rho_0 \cdot \Phi}{\pi a' b' h} ; \tag{12}$$

or, taking

$$\text{grad}_{\rho_0} \left(\frac{1}{2} \rho_0 \cdot \Phi \cdot \rho_0 \right) = \Phi \cdot \rho_0$$

into account and putting

$$\Psi = \frac{1}{2} \frac{fm'}{\pi a' b' h} \rho_0 \cdot \Phi \cdot \rho_0 , \tag{13a}$$

we can write

$$\mathbf{F}_0 = \text{grad}_{\rho_0} \Psi \cdot \tag{13b}$$

By substituting

$$\rho = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} ,$$

$$\rho \times d\rho = (y\,dz - z\,dy)\,\mathbf{i} + (z\,dx - x\,dz)\,\mathbf{j} + (x\,dy - y\,dx)\,\mathbf{k} ,$$

into Equation 8, the following expressions for the components of Φ are obtained:

$$\Phi_{11} = \frac{1}{2}\oint \frac{x(y\,dz - z\,dy)}{\rho^3} ; \tag{14}$$

$$\Phi_{22} = \frac{1}{2}\oint \frac{y(z\,dx - x\,dz)}{\rho^3} ; \tag{15}$$

$$\Phi_{33} = \frac{1}{2}\oint \frac{z(x\,dy - y\,dx)}{\rho^3} ; \tag{16}$$

$$\Phi_{12} = \Phi_{21} = \frac{1}{2}\oint \frac{x(z\,dx - x\,dz)}{\rho^3} = \frac{1}{2}\oint \frac{y(y\,dz - z\,dy)}{\rho^3} ; \tag{17}$$

$$\Phi_{23} = \Phi_{32} = \frac{1}{2}\oint \frac{y\left(x\,dy - y\,dx\right)}{\rho^3} = \frac{1}{2}\oint \frac{z\left(z\,dx - x\,dz\right)}{\rho^3} \; ; \tag{18}$$

$$\Phi_{31} = \Phi_{13} = \frac{1}{2}\oint \frac{z\left(y\,dz - z\,dy\right)}{\rho^3} = \frac{1}{2}\oint \frac{x\left(x\,dy - y\,dx\right)}{\rho^3} \; ; \tag{19}$$

$$\rho^2 = x^2 + y^2 + z^2 \; .$$

The process of integration is performed in the direction of motion of the disturbing body.

We assume that the original system of coordinates and the system attached to the apex of the cone are both right-handed systems. By rotating the system associated with the apex of the cone it is always possible to reduce the equation of the cone to its normal form

$$\frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} = 0 \tag{20a}$$

and at the same time to cause all points of the ring to have positive z-coordinates. Also, without loss of generality we can assume that p, q, and r in Equation 20a satisfy the conditions $p \leq q < 0 < r$.

The direction of integration in Equations 14-19 is positive in the system defined by the unit vectors $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$. However, it can be positive or negative in the system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of principal axes of the cone. The direction of integration will be positive in both systems if $\mathbf{R}' \cdot \mathbf{k} > 0$ and it will be positive in $(\mathbf{P}', \mathbf{Q}', \mathbf{R}')$ and negative in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ if $\mathbf{R}' \cdot \mathbf{k} < 0$.

Let

$$\mathbf{s} = \alpha \mathbf{P}' + \beta \mathbf{Q}' + \gamma \mathbf{R}' \tag{20b}$$

be the position vector of the apex (of the disturbed body) with respect to the center of the ring. We agreed to choose the direction of $\mathbf{k}$ in such a way that the z-coordinates of points of the portion of the plane limited by the ring will be positive in the system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$; in other words we must have

$$-\gamma \mathbf{R}' \cdot \mathbf{k} > 0 \; .$$

From this we conclude that the direction of integration will be positive in both systems if $\gamma < 0$ and it will be positive in the system $(\mathbf{P}', \mathbf{Q}', \mathbf{R}')$ and negative in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ if $\gamma > 0$. A simple geometrical drawing will confirm this fact. The direction of integration can always be taken as positive in the system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ if the factor

$$-\frac{|\gamma|}{\gamma} = -\operatorname{sign}\gamma$$

is attached to the integral. We shall combine this factor with the factor $fm'/\pi a'b'h$ and we shall postpone its introduction until the development is completed. In order to investigate the form of Φ with respect to the system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ let us choose as the contour of integration the ellipse

$$\left.\begin{aligned} \frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} &= 0 \\ z &= 1 . \end{aligned}\right\} \qquad (21)$$

With the condition imposed by Equation 21 the preceding Equations 14-19 become

$$\Phi_{11} = -\frac{1}{2}\oint \frac{x\,dy}{\rho^3} , \quad \Phi_{22} = +\frac{1}{2}\oint \frac{y\,dx}{\rho^3} , \quad \Phi_{33} = +\frac{1}{2}\oint \frac{x\,dy - y\,dx}{\rho^3} , \qquad (22)$$

$$\Phi_{12} = +\frac{1}{2}\oint \frac{x\,dx}{\rho^3} , \quad \Phi_{23} = +\frac{1}{2}\oint \frac{dx}{\rho^3} , \quad \Phi_{31} = -\frac{1}{2}\oint \frac{dy}{\rho^3} . \qquad (23)$$

We have, in the different quadrants of the ellipse given by Equations 21, the following:

	x	y	dx	dy	− x dy	+ y dx	x dy − y dx	x dx
I	+	+	−	+	−	−	+	−
II	−	+	−	−	−	−	+	+
III	−	−	+	−	−	−	+	−
IV	+	−	+	+	−	−	+	+

From this table and Equations 22, we see that

$$\Phi_{11} < 0 , \quad \Phi_{22} < 0 , \quad \Phi_{33} > 0$$

and, because of the symmetry of the contour of integration, each of the integrals in Equations 22 is equal to four times the integral taken over the first quarter of the ellipse (Equations 21). Combining in Equations 23 the elements which are symmetrical with respect to the x and y axes, we also conclude from the above table that

$$\Phi_{12} = \Phi_{23} = \Phi_{31} = 0 ,$$

and Φ takes its normal form

$$\Phi = \Phi_{11}\,\mathbf{i}\,\mathbf{i} + \Phi_{22}\,\mathbf{j}\,\mathbf{j} + \Phi_{33}\,\mathbf{k}\,\mathbf{k}$$

in the system of the principal axes of the cone. (Vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the principal axes of the cone.) In other words, both the potential

$$\Psi = \frac{fm'}{\pi a' b' h} \cdot \frac{1}{2} \boldsymbol{\rho}_0 \cdot \boldsymbol{\Phi} \cdot \boldsymbol{\rho}_0 \tag{13a}$$

and the equation of the cone can be reduced to their normal forms simultaneously.

In order to compute Φ_{11}, Φ_{22}, Φ_{33} in the system defined by the principal axes, we can take the curve defined by the equations

$$\frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} = 0 , \quad x^2 + y^2 + z^2 = 1 \tag{24}$$

as the integration contour. Taking the identities

$$\begin{aligned} (s-p)(q-r) + (s-q)(r-p) + (s-r)(p-q) &= 0 , \\ p(s-p)(q-r) + q(s-q)(r-p) + r(s-r)(p-q) &= (p-q)(q-r)(r-p) , \end{aligned} \tag{25}$$

into account, we can write the equations of the curve defined by Equations 24 in the parametric form:

$$x^2 = \frac{p(s-p)}{(r-p)(p-q)} , \tag{26}$$

$$y^2 = \frac{q(s-q)}{(p-q)(q-r)} , \tag{27}$$

$$z^2 = \frac{r(s-r)}{(q-r)(r-p)} , \tag{28}$$

where s is a variable parameter. Considering the inequalities

$$p < 0 , \quad q < 0 , \quad r > 0 ,$$

$$p - q \leqq 0 , \quad q - r < 0 , \quad r - p > 0 ,$$

$$x^2 \geq 0 , \quad y^2 \geq 0 , \quad z^2 \geq 0 ,$$

we conclude from Equations 26-28 that s must satisfy the conditions

$$p \leq s \leq q , \tag{29}$$

and that the two inequalities

$$p q r > 0 , \quad (p-q)(q-r)(r-p) > 0 \tag{30}$$

are satisfied for p, q, and r.

Taking into account the symmetrical form with respect to the z-axis of the contour of integration and the fact that $\rho = 1$ on this contour, we can write Equations 14-16 in the form:

$$\Phi_{11} = 2 \int x y z \left[d(\log z) - d(\log y) \right] , \tag{31}$$

$$\Phi_{22} = 2 \int x y z \left[d(\log x) - d(\log z) \right] , \tag{32}$$

$$\Phi_{33} = 2 \int x y z \left[d(\log y) - d(\log x) \right] . \tag{33}$$

Now the integral is taken along the first quadrant of the contour. On the first quadrant we have $y = 0$ for $s = q$ and $x = 0$ for $s = p$; and the integration is performed in the positive direction from the point

$$x = + \sqrt{\frac{p}{p-r}} , \quad y = 0 , \quad z = + \sqrt{\frac{r}{r-p}} , \quad \text{or} \quad s = q ,$$

to the point

$$x = 0 , \quad y = + \sqrt{\frac{q}{q-r}} \quad z = + \sqrt{\frac{r}{r-q}} , \quad \text{or} \quad s = p .$$

We deduce from Equations 26-28:

$$d(\log x) = \frac{1}{2} \frac{ds}{s-p} , \tag{34}$$

$$d(\log y) = \frac{1}{2} \frac{ds}{s-q} , \tag{35}$$

$$d(\log z) = \frac{1}{2} \frac{ds}{s-r} . \tag{36}$$

In the first quadrant of the contour, we have

$$x y z = - \frac{\sqrt{p q r}}{(p-q)(q-r)(r-p)} \sqrt{(s-p)(s-q)(s-r)} . \tag{37}$$

When Equations 34-37 are taken into consideration, Equations 31-33 become:

$$\Phi_{11} = -\int_q^p \frac{\sqrt{pqr}}{(p-q)(r-p)} \sqrt{(s-p)(s-q)(s-r)} \cdot \frac{ds}{(s-q)(s-r)} \tag{38}$$

$$\Phi_{22} = -\int_q^p \frac{\sqrt{pqr}}{(p-q)(q-r)} \sqrt{(s-p)(s-q)(s-r)} \cdot \frac{ds}{(s-r)(s-p)} \tag{39}$$

$$\Phi_{33} = -\int_q^p \frac{\sqrt{pqr}}{(q-r)(r-p)} \sqrt{(s-p)(s-q)(s-r)} \cdot \frac{ds}{(s-p)(s-q)} . \tag{40}$$

The sign of the square root is chosen to be negative in order that the conditions $\Phi_{11} < 0$, $\Phi_{22} < 0$, $\Phi_{33} > 0$ be satisfied. As before, the integration is performed in the positive direction along the first quarter of the contour. Putting

$$\left.\begin{aligned} e_1 &= \frac{1}{3}(2r-p-q) = r - \frac{1}{3}(p+q+r) , \\ e_2 &= \frac{1}{3}(2q-r-p) = q - \frac{1}{3}(p+q+r) , \\ e_3 &= \frac{1}{3}(2p-q-r) = p - \frac{1}{3}(p+q+r) , \end{aligned}\right\} \tag{41}$$

we have $e_1 + e_2 + e_3 = 0$. From $p \leqq q < r$ and Equations 41, we obtain

$$e_1 > e_2 \geq e_3 .$$

Let us now introduce in place of s a new independent variable u, by means of the equation

$$s = p(u) + \frac{1}{3}(p+q+r) , \tag{42}$$

where $p(u)$ is the Weierstrass elliptic function satisfying the equation

$$p'^2(u) = 4\left(p(u) - e_1\right)\left(p(u) - e_2\right)\left(p(u) - e_3\right) . \tag{43}$$

Equations 41-43 now become

$$s - p = p(u) - e_3 , \tag{44}$$

$$s - q = p(u) - e_2 , \tag{45}$$

$$s - r = p(u) - e_1 , \tag{46}$$

$$ds = p'(u)\,du , \tag{47}$$

$$(s-p)(s-q)(s-r) = \frac{1}{4} p'^2(u) .$$

The process of integration in Equations 38-40 is performed in the positive direction from $s = q$ to $s = p$. We have for $s = q$

$$p(u) = e_2 ,$$

and for $s = p$

$$p(u) = e_3 .$$

Consequently in the first quarter of the contour the p-function is decreasing and $p'(u) < 0$, or

$$\sqrt{(s-p)(s-q)(s-r)} = -\frac{1}{2} p'(u) \tag{48}$$

Substituting Equations 44-48 into Equations 38-40, we deduce

$$\Phi_{11} = \frac{1}{2} \int \frac{\sqrt{pqr}}{(p-q)(r-p)} \cdot \frac{p'^2(u)\,du}{\left(p(u)-e_2\right)\left(p(u)-e_1\right)} , \tag{49}$$

$$\Phi_{22} = \frac{1}{2} \int \frac{\sqrt{pqr}}{(p-q)(q-r)} \cdot \frac{p'^2(u)\,du}{\left(p(u)-e_3\right)\left(p(u)-e_1\right)} , \tag{50}$$

$$\Phi_{33} = \frac{1}{2} \int \frac{\sqrt{pqr}}{(q-r)(r-p)} \cdot \frac{p'^2(u)\,du}{\left(p(u)-e_3\right)\left(p(u)-e_2\right)} , \tag{51}$$

where the integration is still performed in the positive direction over the first quarter of the contour. Substituting the value of $p'^2(u)$ given by Equation 43, we obtain:

$$\frac{1}{2}\Phi_{11} = \int C(q-r)\left[p(u)-e_3\right] du , \tag{52a}$$

$$\frac{1}{2}\Phi_{22} = \int C(r-p)\left[p(u)-e_2\right] du , \tag{53a}$$

$$\frac{1}{2}\Phi_{33} = \int C(p-q)\left[p(u)-e_1\right] du , \tag{54a}$$

where

$$C = \frac{\sqrt{pqr}}{(p-q)\,(q-r)\,(r-p)} \ ,$$

and the path of integration is the same as for Equations 49-51. The complete contour is described if the parameter u varies from a certain given value $u = u_0$ to $u = u_0 + 4\omega$, where 2ω is the real period of $p(u)$:

$$\omega = \int_{e_3}^{e_2} \frac{dx}{\sqrt{4(x-e_1)\,(x-e_2)\,(x-e_3)}} \ .$$

Taking into account the fact that in our case the integral over the first quarter is equal to one fourth of the integral taken over the whole contour, we have from Equations 52a, 53a, and 54a:

$$2\Phi_{11} = \int_{u_0}^{u_0+4\omega} C\,(q-r)\left[p(u)-e_3\right] du \ , \tag{52b}$$

$$2\Phi_{22} = \int_{u_0}^{u_0+4\omega} C\,(r-p)\left[p(u)-e_2\right] du \ , \tag{53b}$$

$$2\Phi_{33} = \int_{u_0}^{u_0+4\omega} C\,(p-q)\left[p(u)-e_1\right] du \ . \tag{54b}$$

Taking into account the relations

$$p(u) = -\zeta'(u) \ ,$$

$$\zeta(u+4\omega) - \zeta(u) = 4\eta \ ,$$

$$\eta = \zeta(\omega) \ ,$$

between Weierstrass functions, we deduce from Equations 52b, 53b, and 54b:

$$\frac{1}{2}\Phi_{11} = -C\,(q-r)\left(\eta+\omega e_3\right) \ , \tag{55}$$

$$\frac{1}{2}\Phi_{22} = -C\,(r-p)\left(\eta+\omega e_2\right) \ , \tag{56}$$

$$\frac{1}{2}\Phi_{33} = -C\,(p-q)\left(\eta+\omega e_1\right) \ . \tag{57}$$

Eliminating e_1, e_2, e_3 from the previous equations by means of Equations 41, we obtain:

$$\frac{1}{2}\Phi_{11} = -C(q-r)\left[\eta + \left(p - \frac{p+q+r}{3}\right)\omega\right], \tag{58}$$

$$\frac{1}{2}\Phi_{22} = -C(r-p)\left[\eta + \left(q - \frac{p+q+r}{3}\right)\omega\right], \tag{59}$$

$$\frac{1}{2}\Phi_{33} = -C(p-q)\left[\eta + \left(r - \frac{p+q+r}{3}\right)\omega\right]. \tag{60}$$

Putting

$$M = C\left[(q-r)\,\mathbf{i\,i} + (r-p)\,\mathbf{j\,j} + (p-q)\,\mathbf{k\,k}\right] \tag{61}$$

and

$$N = C\left[(q-r)\,p\,\mathbf{i\,i} + (r-p)\,q\,\mathbf{j\,j} + (p-q)\,r\,\mathbf{k\,k}\right], \tag{62}$$

we have, from Equations 58-60,

$$-\frac{1}{2}\Phi = M\eta + N\omega - \frac{p+q+r}{3}M\omega \,. \tag{63}$$

The next step will be to express Φ in terms of the fundamental dyadic

$$\Theta = \frac{\mathbf{i\,i}}{p} + \frac{\mathbf{j\,j}}{q} + \frac{\mathbf{k\,k}}{r} \tag{64}$$

and in terms of its invariants

$$k_1 = p + q + r, \tag{65}$$

$$k_2 = pq + qr + rp, \tag{66}$$

$$k_3 = pqr. \tag{67}$$

The dyadic Θ is closely associated with the cone; its components in the inertial system and its invariants (in terms of these components) can be easily deduced. If the dyadics Φ, M, N are expressed in terms of Θ and its invariants, then the components of Φ, M and N in the inertial system can be obtained with no difficulty. Our final goal in this section will be to obtain an expression for Φ and F_0 in terms of invariants of the cone, and in the system defined by the unit vectors P', Q', R'.

The dyadic M can be written in the form of a determinant with dyadical elements:

$$M = C \begin{vmatrix} \mathbf{i\,i} & \mathbf{j\,j} & \mathbf{k\,k} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} . \tag{68}$$

Putting

$$\Lambda = \Theta^{-1} = p\,\mathbf{i\,i} + q\,\mathbf{j\,j} + r\,\mathbf{k\,k} ,$$

we deduce from Equation 68 that

$$M \cdot \begin{vmatrix} p & q & r \\ 1 & 1 & 1 \\ \frac{1}{p} & \frac{1}{q} & \frac{1}{r} \end{vmatrix} = C \begin{vmatrix} \Lambda & I & \Theta \\ k_1^2 - 2k_2 & k_1 & 3 \\ k_1 & 3 & \frac{k_2}{k_3} \end{vmatrix} , \tag{69}$$

where I is the idemfactor

$$I = \mathbf{i\,i} + \mathbf{j\,j} + \mathbf{k\,k} .$$

Taking into account that

$$\begin{vmatrix} p & q & r \\ 1 & 1 & 1 \\ \frac{1}{p} & \frac{1}{q} & \frac{1}{r} \end{vmatrix} = - \frac{(p-q)\,(q-r)\,(r-p)}{k_3} ,$$

and also that

$$C = \frac{\sqrt{k_3}}{(p-q)\,(q-r)\,(r-p)} , \tag{70}$$

we deduce from Equation 69 that

$$M = -\frac{\sqrt{k_3}}{(p-q)^2\,(q-r)^2\,(r-p)^2}\cdot\begin{vmatrix} \Lambda & I & \Theta \\ k_1^2-2k_2 & k_1 & 3 \\ k_1k_3 & 3k_3 & k_2 \end{vmatrix}. \tag{71}$$

However, it follows from Equation 41 that

$$e_3 - e_2 = p - q\,,$$

$$e_1 - e_3 = r - p\,,$$

$$e_2 - e_1 = q - r\,,$$

and thus Equation 71 becomes

$$M = -\frac{\sqrt{k_3}}{(e_1-e_2)^2\,(e_2-e_3)^2\,(e_3-e_1)^2}\cdot\begin{vmatrix} \Lambda & I & \Theta \\ k_1^2-2k_2 & k_1 & 3 \\ k_1k_3 & 3k_3 & k_2 \end{vmatrix}. \tag{72}$$

The expression $(e_1-e_2)^2\,(e_2-e_3)^2\,(e_3-e_1)^2$ is the discriminant of the equation

$$4\,(x-e_1)\,(x-e_2)\,(x-e_3) = 4x^3 - g_2x - g_3 = 0\,,$$

where

$$-g_2 = 4(e_1e_2+e_2e_3+e_3e_1)\,,$$

$$g_3 = 4e_1e_2e_3\,,$$

and g_2 and g_3 are expressible as invariants of the dyadic Θ.

We know, from either the theory of elliptic functions or the theory of equations, that

$$(p-q)^2\,(q-r)^2\,(r-p)^2 = (e_1-e_2)^2\,(e_2-e_3)^2\,(e_3-e_1)^2 = \frac{1}{16}\Delta\,, \tag{73}$$

where

$$\Delta = g_2^3 - 27 g_3^2 . \tag{74}$$

It follows from Equation 72, by taking the form of the discriminant (Equation 73) into consideration, that

$$\begin{aligned} \frac{\Delta}{16\sqrt{k_3}} M &= \left(9k_3 - k_1 k_2\right) \Lambda \\ &+ \left(k_1^2 k_2 - 2k_2^2 - 3k_1 k_3\right) I \\ &+ 2\left(3k_2 - k_1^2\right) k_3 \Theta . \end{aligned} \tag{75}$$

Thus the dyadic M is expressible as a linear combination of dyadics $\Lambda = \Theta^{-1}$, I, and Θ. The left side of Equation 75a will be designated by μ in accordance with Halphen's notations:

$$\mu = \frac{\Delta}{16\sqrt{k_3}} M . \tag{75b}$$

The dyadic N (Equation 62), like M, can also be written in the form of a determinant with dyadical elements in the first row. From Equation 62 we have

$$\frac{N}{C} = \begin{vmatrix} p\,i\,i & q\,j\,j & r\,k\,k \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} . \tag{76}$$

We deduce from Equation 76 that

$$\frac{N}{C} \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{p} & \frac{1}{q} & \frac{1}{r} \\ \frac{q-r}{p} & \frac{r-p}{q} & \frac{p-q}{r} \end{vmatrix} = \begin{vmatrix} \Lambda & I & \frac{M}{C} \\ k_1 & 3 & 0 \\ 3 & \frac{k_2}{k_3} & -\frac{(p-q)(q-r)(r-p)}{k_3} \end{vmatrix} ; \tag{77}$$

and from Equations 73 and 75b we obtain

$$\frac{M}{C} = \frac{16\sqrt{k_3}}{\Delta} \cdot \frac{\mu}{C} = \frac{\mu}{(p-q)(q-r)(r-p)} . \tag{78}$$

The value of the determinant in the left side of Equation 77 is

$$\frac{2}{pqr}\cdot\left[\left(p^2+q^2+r^2\right)-(pq+qr+rp)\right]$$

$$=\frac{2}{pqr}\cdot\left[(p+q+r)^2-3(pq+qr+rp)\right]$$

$$=\frac{2\left(k_1^2-3k_2\right)}{k_3},$$

as can be shown very easily. Substituting the above value into Equation 77 and taking Equations 73 and 78 into account, we have

$$\frac{2\left(k_1^2-3k_2\right)}{k_3}\cdot\frac{N}{C}=\frac{1}{(p-q)\ (q-r)\ (r-p)}\begin{vmatrix}\Lambda & I & \mu\\ k_1 & 3 & 0\\ 3 & \frac{k_2}{k_3} & -\frac{1}{16}\frac{\Delta}{k_3}\end{vmatrix}. \tag{79}$$

By taking $1/C$ in the form (from Equations 70 and 73)

$$\frac{1}{C}=\frac{1}{16}\cdot\frac{\Delta}{(p-q)\ (q-r)\ (r-p)\sqrt{k_3}},$$

we finally obtain from Equation 79

$$\frac{\Delta}{16\sqrt{k_3}}N=\frac{1}{2\left(k_1^2-3k_2\right)}\cdot\left[\frac{1}{16}\Delta\left(k_1 I-3\Lambda\right)+\left(k_1k_2-9k_3\right)\mu\right]. \tag{80}$$

In proving Equations 75a and 80 we followed Halphen's reasoning closely, using dyadics instead of quadratic forms. We have shown that the dyadic N, like M, is a linear combination of three fundamental dyadics (Λ, I, Θ) with the coefficients dependent upon the invariants of Θ.

In our proof we used the system of principal axes of the cone, but the use of dyadics leads immediately to the conclusion that the properties of M and N expressed by means of Equations 75a and 80 exist in *any* system of coordinates.

Equations 75a and 80 can be written in the form:

$$M=\frac{16\sqrt{k_3}}{\Delta}\left[\left(9k_3-k_1k_2\right)\Lambda+\left(k_1^2k_2-2k_2^2-3k_1k_3\right)I\right.$$
$$\left.+\,2k_3\left(3k_2-k_1^2\right)\Theta\right], \tag{81}$$

$$N = \frac{1}{2(k_1^2 - 3k_2)} \left[(k_1 I - 3\Lambda)\sqrt{k_3} + (k_1 k_2 - 9k_3) M \right], \tag{82}$$

where, as before,

$$\frac{\Delta}{16} = (e_1 - e_2)^2 (e_2 - e_3)^2 (e_3 - e_1)^2$$

$$= (p - q)^2 (q - r)^2 (r - p)^2$$

$$= \frac{1}{16}(g_2^3 - 27 g_3^2) .$$

Now, the discriminant of any cubic equation of the form

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

can be written as

$$a_1^2 a_2^2 - 4 a_1^3 a_3 + 18 a_1 a_2 a_3 - 4 a_2^3 - 27 a_3^2 .$$

For the equation

$$x^3 - k_1 x^2 + k_2 x - k_3 = 0 ,$$

whose roots are the reciprocals of the characteristic roots $1/p$, $1/q$, $1/r$, for the cone, we deduce that

$$\frac{\Delta}{16} = k_1^2 k_2^2 - 4 k_1^3 k_3 + 18 k_1 k_2 k_3 - 4 k_2^3 - 27 k_3^2 \tag{83}$$

In addition, from Equations 41 and 65-67, we have

$$-\frac{1}{4} g_2 = \left(p - \frac{1}{3} k_1\right)\left(q - \frac{1}{3} k_1\right) + \left(q - \frac{1}{3} k_1\right)\left(r - \frac{1}{3} k_1\right) + \left(r - \frac{1}{3} k_1\right)\left(p - \frac{1}{3} k_1\right),$$

$$-\frac{1}{4} g_3 = \left(\frac{1}{3} k_1 - p\right)\left(\frac{1}{3} k_1 - q\right)\left(\frac{1}{3} k_1 - r\right) ;$$

and, after some transformations,

$$g_2 = \frac{4}{3}(k_1^2 - 3 k_2) , \tag{84}$$

$$g_3 = \frac{4}{27}(2 k_1^3 - 9 k_1 k_2 + 27 k_3) . \tag{85}$$

The next step will be to obtain an expression for Φ in terms of the invariants of the cone.

We deduce from Equations 63, 82, 84 and 85 that

$$-\frac{1}{2}\Phi = M\left(\eta - \frac{3}{2}\frac{g_3}{g_2}\omega\right) + \frac{2}{3}\omega\frac{\sqrt{k_3}}{g_2}\left(k_1 I - 3\Lambda\right) . \tag{86}$$

A second form of Equation 86 that will be useful in further exposition is obtained by eliminating M in favor of μ:

$$\frac{1}{2}\Phi = -\left[\frac{16\mu\sqrt{k_3}}{\Delta}\left(\eta - \frac{3}{2}\frac{g_3}{g_2}\omega\right) + \frac{2}{3}\frac{\sqrt{k_3}}{g_2}\left(k_1 I - 3\Lambda\right)\omega\right] . \tag{87}$$

At this point it will be convenient to attach to Φ the factor

$$-\frac{fm'\,|\gamma|}{\pi a' b' h \gamma}$$

which was purposely omitted throughout the exposition starting from Equation 22. Taking (Equation 20b)

$$h = \mathbf{p}_0 \cdot \mathbf{R}' = -\mathbf{s} \cdot \mathbf{R}' = -\left(\alpha\mathbf{P}' + \beta\mathbf{Q}' + \gamma\mathbf{R}'\right) \cdot \mathbf{R}' = -\gamma$$

into account and defining

$$A = \frac{16}{\pi\Delta}\left(\frac{3}{2}\frac{g_3}{g_2}\omega - \eta\right) , \tag{88}$$

$$B = \frac{2\omega}{\pi g_2} , \tag{89}$$

we obtain from Equation 87:

$$\Phi = \frac{2\,fm'\sqrt{k_3}}{a' b' |\gamma|}\left[\mu A - \frac{1}{3}B\left(k_1 I - 3\Lambda\right)\right] . \tag{90}$$

Now we can express the disturbing force $\mathbf{F}_0$, averaged over the orbit of the disturbing body, by the formula

$$\mathbf{F}_0 = \Phi \cdot \mathbf{p}_0$$

or

$$\mathbf{F}_0 = -\Phi \cdot \mathbf{r} , \tag{91}$$

where Φ is now given by Equation 90.

In a following chapter we shall derive expressions for the coefficients A and B in terms of invariants of Θ.

ON THE FORM OF THE BASIC DYADIC Φ IN TERMS OF INVARIANTS OF Θ

Let us choose the system (P', Q', R'), with origin in the center of the ring, as a basic reference system. Let (α, β, γ), as before, be the coordinates of the apex of the cone, (ξ, η, ζ) be the coordinates of a point of the cone and $(x_0, y_0, 0)$ be the coordinates of the intersection of the generating line, passing through (ξ, η, ζ), with the ring. Thus we have

$$\frac{\xi - \alpha}{\alpha - x_0} = \frac{\eta - \beta}{\beta - y_0} = \frac{\zeta - \gamma}{\gamma} \tag{92}$$

and

$$\frac{x_0^2}{a'^2} + \frac{y_0^2}{b'^2} = 1 . \tag{93}$$

If the origin of the coordinates is transferred to the apex, then the coordinates of a point of the cone become

$$x = \xi - \alpha , \quad y = \eta - \beta , \quad z = \zeta - \gamma$$

and from Equation 92 we have

$$x_0 = \frac{\alpha z - \gamma x}{z} \quad \text{and} \quad y_0 = \frac{\beta z - \gamma y}{z} .$$

Substituting these values into Equation 93, we obtain the equation of the cone in the form

$$\frac{z^2}{\gamma^2} - \frac{(\alpha z - \gamma x)^2}{a'^2 \gamma^2} - \frac{(\beta z - \gamma y)^2}{b'^2 \gamma^2} = 0 . \tag{94}$$

The divisor γ^2 is introduced for reasons of homogeneity and the condition

$$p \leq q < r$$

requires an arrangement of signs as in Equation 94. Therefore this equation can also be written in the form:

$$\mathbf{w} \cdot \Theta \cdot \mathbf{w} = 0 ,$$

where

$$\mathbf{w} = x\mathbf{P}' + y\mathbf{Q}' + z\mathbf{R}' \tag{95}$$

and

$$\Theta = \frac{\mathbf{R}'\mathbf{R}'}{\gamma^2} - \frac{(\alpha\mathbf{R}' - \gamma\mathbf{P}')\ (\alpha\mathbf{R}' - \gamma\mathbf{P}')}{a'^2\gamma^2} - \frac{(\beta\mathbf{R}' - \gamma\mathbf{Q}')\ (\beta\mathbf{R}' - \gamma\mathbf{Q}')}{b'^2\gamma^2} . \tag{96}$$

By again introducing $\mathbf{s}$, the position vector of the apex with respect to the center of the ring

$$\mathbf{s} = \alpha\mathbf{P}' + \beta\mathbf{Q}' + \gamma\mathbf{R}' , \tag{20b}$$

we can write the dyadic Θ (Equation 96) in a contracted form:

$$\Theta = \frac{\mathbf{s} \times \mathbf{Q}'\mathbf{Q}' \times \mathbf{s}}{a'^2\gamma^2} + \frac{\mathbf{s} \times \mathbf{P}'\mathbf{P}' \times \mathbf{s}}{b'^2\gamma^2} + \frac{\mathbf{R}'\mathbf{R}'}{\gamma^2} , \tag{97}$$

which is more convenient for the computation of invariants.

The form of Θ which we have used previously was

$$\Theta = \frac{\mathbf{i}\,\mathbf{i}}{p} + \frac{\mathbf{j}\,\mathbf{j}}{q} + \frac{\mathbf{k}\,\mathbf{k}}{r} \tag{64}$$

and it referred to the principal axis of the cone.

In the process of computing the invariants we will follow the classical Gibbs notations (Reference 7) closely.

If a dyadic X is given in a reduced form,

$$X = \mathbf{a}\,\mathbf{l} + \mathbf{b}\,\mathbf{m} + \mathbf{c}\,\mathbf{n} , \tag{98}$$

then the adjoined dyadic X_2 and the invariants X_s, $(X_2)_s$, X_3 are given by the formulas (Reference 7):

$$X_2 = \mathbf{b} \times \mathbf{c}\,\mathbf{m} \times \mathbf{n} + \mathbf{c} \times \mathbf{a}\,\mathbf{n} \times \mathbf{l} + \mathbf{a} \times \mathbf{b}\,\mathbf{l} \times \mathbf{m} , \tag{99}$$

$$X_s = \mathbf{a} \cdot \mathbf{l} + \mathbf{b} \cdot \mathbf{m} + \mathbf{c} \cdot \mathbf{n} \,, \tag{100}$$

$$(X_2)_s = \mathbf{b} \times \mathbf{c} \cdot \mathbf{m} \times \mathbf{n} + \mathbf{c} \times \mathbf{a} \cdot \mathbf{n} \times \mathbf{l} + \mathbf{a} \times \mathbf{b} \cdot \mathbf{l} \times \mathbf{m} \,, \tag{101}$$

$$X_3 = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\,(\mathbf{l} \cdot \mathbf{m} \times \mathbf{n}) \,. \tag{102}$$

The reciprocal dyadic is given by the formula:

$$X^{-1} = \frac{X_2}{X_3} \,. \tag{103}$$

Substituting into Equations 99-103 first

$$\mathbf{a} = \frac{\mathbf{s} \times \mathbf{Q}'}{a'^2 \gamma^2} \,, \quad \mathbf{b} = \frac{\mathbf{s} \times \mathbf{P}'}{b'^2 \gamma^2} \,, \quad \mathbf{c} = \frac{\mathbf{R}'}{\gamma^2} \,,$$

$$\mathbf{l} = \mathbf{Q}' \times \mathbf{s} \,, \quad \mathbf{m} = \mathbf{P}' \times \mathbf{s} \,, \quad \mathbf{n} = \mathbf{R}' \,,$$

and then

$$\mathbf{a} = \frac{\mathbf{i}}{p} \,, \quad \mathbf{b} = \frac{\mathbf{j}}{q} \,, \quad \mathbf{c} = \frac{\mathbf{k}}{r} \,,$$

$$\mathbf{l} = \mathbf{i} \,, \quad \mathbf{m} = \mathbf{j} \,, \quad \mathbf{n} = \mathbf{k} \,,$$

we compute $\Theta_2, \Theta_s, (\Theta_2)_s, \Theta_3$, and Θ^{-1} using both forms of Θ as given by Equations 97 and 64, and compare the results. After the substitution and some easy vectorial transformations based on

$$\left.\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}\,\mathbf{a} \cdot \mathbf{c} - \mathbf{c}\,\mathbf{a} \cdot \mathbf{b} \\ (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \mathbf{a} \cdot \mathbf{c}\,\mathbf{b} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{d}\,\mathbf{b} \cdot \mathbf{c} \end{aligned}\right\} \tag{104}$$

are performed, we obtain:

$$\Theta_2 = -\frac{\mathbf{P}'\,\mathbf{P}'}{b'^2 \gamma^2} - \frac{\mathbf{Q}'\,\mathbf{Q}'}{a'^2 \gamma^2} + \frac{\mathbf{s}\,\mathbf{s}}{a'^2 b'^2 \gamma^2} = \frac{\mathbf{i}\,\mathbf{i}}{q\,r} + \frac{\mathbf{j}\,\mathbf{j}}{r\,p} + \frac{\mathbf{k}\,\mathbf{k}}{p\,q} \,,$$

$$\Theta_s = \frac{1}{a'^2 b'^2 \gamma^2} \left[a'^2 b'^2 - \alpha^2 b'^2 - \beta^2 a'^2 - \gamma^2 \left(a'^2 + b'^2 \right) \right] = \frac{k_2}{k_3} \,,$$

$$\left(\Theta_2\right)_s = -\frac{1}{b'^2\gamma^2} - \frac{1}{a'^2\gamma^2} + \frac{\alpha^2+\beta^2+\gamma^2}{a'^2 b'^2\gamma^2} = \frac{k_1}{k_3}, \tag{107}$$

$$\Theta_3 = \frac{1}{a'^2 b'^2\gamma^2} = \frac{1}{k_3}, \tag{108}$$

$$\Lambda = \Theta^{-1} = \mathbf{s}\,\mathbf{s} - a'^2\,\mathbf{P}'\,\mathbf{P}' - b'^2\,\mathbf{Q}'\,\mathbf{Q}' . \tag{109}$$

From Equations 106-108 we deduce expressions for the invariants k_1, k_2, k_3:

$$k_1 = \alpha^2+\beta^2+\gamma^2 - \left(a'^2+b'^2\right), \tag{110}$$

$$k_2 = a'^2b'^2 - \alpha^2 b'^2 - \beta^2 a'^2 - \gamma^2\left(a'^2+b'^2\right), \tag{111}$$

$$k_3 = a'^2b'^2\gamma^2 . \tag{112}$$

The value $\sqrt{k_3}$ in Equation 90 is positive. Consequently, from Equation 112,

$$\sqrt{k_3} = a'\,b'\,|\gamma| .$$

Substituting this value into Equation 90, we obtain a final form of Φ:

$$\Phi = 2\,fm'\,(A\mu + B\nu), \tag{113}$$

where

$$\nu = \Lambda - \frac{1}{3}k_1\,I . \tag{114a}$$

The expression (Equation 91) for the disturbing force averaged over the orbit of the disturbing body now becomes

$$\mathbf{F}_0 = -\,2\,fm'\,(A\mu\cdot\mathbf{r} + B\nu\cdot\mathbf{r}) . \tag{114b}$$

Using Equation 96 we can write Θ in the form

$$\begin{aligned} k_3\,\Theta = &-b'^2\gamma^2\,\mathbf{P}'\,\mathbf{P}' && + b'^2\alpha\gamma\,\mathbf{P}'\,\mathbf{R}' \\ &-a'^2\gamma^2\,\mathbf{Q}'\,\mathbf{Q}' && + a'^2\beta\gamma\,\mathbf{Q}'\,\mathbf{R}' \\ &+ b'^2\alpha\gamma\,\mathbf{R}'\,\mathbf{P}' + a'^2\gamma\beta\,\mathbf{R}'\,\mathbf{Q}' && + \left(a'^2b'^2 - \alpha^2 b'^2 - \beta^2 a'^2\right)\mathbf{R}'\,\mathbf{R}' . \end{aligned} \tag{115}$$

In a similar way, from Equation 109 we obtain

$$\begin{aligned}\Lambda = \left(\alpha^2 - a'^2\right) P' P' &+ \alpha\beta\, P' Q' + \alpha\gamma\, P' R' \\ + \alpha\beta\, Q' P' &+ \left(\beta^2 - b'^2\right) Q' Q' + \beta\gamma\, Q' R' \\ + \alpha\gamma\, R' P' &+ \gamma\beta\, R' Q' + \gamma^2\, R' R' ;\end{aligned} \tag{116}$$

and in addition we have

$$I = P' P' + Q' Q' + R' R' . \tag{117}$$

Now, substituting the results given by Equations 115-117 into 75a and 114a, we obtain μ and ν in the form of Halphen's matrices

$$\mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}, \tag{118}$$

$$\nu = \begin{bmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \nu_{31} & \nu_{32} & \nu_{33} \end{bmatrix}, \tag{119}$$

where

$$\begin{aligned}
\mu_{11} &= \left(9k_3 - k_1 k_2\right)\left(\alpha^2 - a'^2\right) + k_1\left(k_1 k_2 - 3k_3\right) - 2k_2^2 + \frac{3}{2} g_2\, b'^2 \gamma^2 , \\
\mu_{22} &= \left(9k_3 - k_1 k_2\right)\left(\beta^2 - b'^2\right) + k_1\left(k_1 k_2 - 3k_3\right) - 2k_2^2 + \frac{3}{2} g_2\, a'^2 \gamma^2 , \\
\mu_{33} &= \left(9k_3 - k_1 k_2\right)\gamma^2 + k_1\left(k_1 k_2 - 3k_3\right) - 2k_2^2 + \frac{3}{2} g_2 \left(\alpha^2 b'^2 + \beta^2 a'^2 - a'^2 b'^2\right) , \\
\mu_{12} &= \mu_{21} = \left(9k_3 - k_1 k_2\right)\alpha\beta , \\
\mu_{23} &= \mu_{32} = \left(9k_3 - k_1 k_2\right)\beta\gamma - \frac{3}{2} g_2\, a'^2 \beta\gamma , \\
\mu_{31} &= \mu_{13} = \left(9k_3 - k_1 k_2\right)\gamma\alpha - \frac{3}{2} g_2\, b'^2 \gamma\alpha ;
\end{aligned} \tag{120}$$

and

$$
\begin{aligned}
\nu_{11} &= \alpha^2 - a'^2 - \frac{1}{3} k_1 , \\
\nu_{22} &= \beta^2 - b'^2 - \frac{1}{3} k_1 , \\
\nu_{33} &= \gamma^2 - \frac{1}{3} k_1 , \\
\nu_{12} &= \nu_{21} = \alpha\beta , \\
\nu_{23} &= \nu_{32} = \beta\gamma , \\
\nu_{31} &= \nu_{13} = \gamma\alpha .
\end{aligned} \tag{121}
$$

The decomposition of Φ in the system $(\mathbf{P}', \mathbf{Q}', \mathbf{R}')$ takes the form

$$\Phi_{ij} = 2\,\mathrm{fm}' \left(A \mu_{ij} + B \nu_{ij} \right). \tag{122}$$

Putting

$$x = \mathbf{r} \cdot \mathbf{P}' , \quad y = \mathbf{r} \cdot \mathbf{Q}' , \quad z = \mathbf{r} \cdot \mathbf{R}' , \tag{123}$$

we obtain a decomposition of $-\mathbf{F}_0$:

$$-F_{01} = \Phi_{11}\, x + \Phi_{12}\, y + \Phi_{13}\, z , \tag{124}$$

$$-F_{02} = \Phi_{21}\, x + \Phi_{22}\, y + \Phi_{23}\, z , \tag{125}$$

$$-F_{03} = \Phi_{31}\, x + \Phi_{32}\, y + \Phi_{33}\, z . \tag{126}$$

The system of Equations 124-126 gives Halphen's decomposition of $\mathbf{F}_0$ along the axes $(\mathbf{P}', \mathbf{Q}', \mathbf{R}')$. Here, however, the author would like to suggest a slightly different system of formulas, which might be simpler to program for computers. This system is based on decompositions of Θ and Λ as given by Equations 97 and 109. Putting

$$
\begin{aligned}
\left(\Theta_1\right) &= k_3 \mathbf{P}' \cdot \Theta \cdot \mathbf{r} , \\
\left(\Theta_2\right) &= k_3 \mathbf{Q}' \cdot \Theta \cdot \mathbf{r} , \\
\left(\Theta_3\right) &= k_3 \mathbf{R}' \cdot \Theta \cdot \mathbf{r} ,
\end{aligned} \tag{127}
$$

$$
\begin{aligned}
(\Lambda_1) &= \mathbf{P}' \cdot \Lambda \cdot \mathbf{r}, \\
(\Lambda_2) &= \mathbf{Q}' \cdot \Lambda \cdot \mathbf{r}, \\
(\Lambda_3) &= \mathbf{R}' \cdot \Lambda \cdot \mathbf{r},
\end{aligned}
\tag{128}
$$

$$
\begin{aligned}
x &= \mathbf{P}' \cdot \mathbf{r}, \\
y &= \mathbf{Q}' \cdot \mathbf{r}, \\
z &= \mathbf{R}' \cdot \mathbf{r},
\end{aligned}
$$

and taking the equations

$$
\begin{aligned}
\mathbf{s} &= \mathbf{r} + a' e' \mathbf{P}', \\
\mathbf{s} \times \mathbf{r} &= a' e' \mathbf{P}' \times \mathbf{r}
\end{aligned}
\tag{129}
$$

into account, we deduce from Equations 97 and 109:

$$
\begin{aligned}
(\Theta_1) &= a' b'^2 e' \gamma^2, \\
(\Theta_2) &= 0, \\
(\Theta_3) &= a' b'^2 \gamma (a' - \alpha e'),
\end{aligned}
\tag{130}
$$

$$
\begin{aligned}
(\Lambda_1) &= \alpha \mathbf{s} \cdot \mathbf{r} - a'^2 x, \\
(\Lambda_2) &= \beta \mathbf{s} \cdot \mathbf{r} - b'^2 y, \\
(\Lambda_3) &= \gamma \mathbf{s} \cdot \mathbf{r}.
\end{aligned}
\tag{131}
$$

Now let us introduce two vectors $\mathbf{m}$ and $\mathbf{n}$ by means of the formulas:

$$
\mathbf{m} = \begin{bmatrix} (\Lambda_1) & x & (\Theta_1) \\ (\Lambda_2) & y & (\Theta_2) \\ (\Lambda_3) & z & (\Theta_3) \end{bmatrix} \cdot \begin{bmatrix} 9k_3 - k_1 k_2 \\ k_1 (k_1 k_2 - 3k_3) - 2k_2^2 \\ -\frac{3}{2} g_2 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix},
\tag{132}
$$

$$\mathbf{n} = \begin{bmatrix} (\Lambda_1) - \frac{1}{3} k_1 x \\ (\Lambda_2) - \frac{1}{3} k_1 y \\ (\Lambda_3) - \frac{1}{3} k_1 z \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} . \tag{133}$$

By taking Equations 75a, 75b and 114a into account, the expressions for the components F_{01}, F_{02}, F_{03} take a more concise form:

$$\left.\begin{aligned} F_{01} &= -2\,fm'\left(A m_1 + B n_1\right) , \\ F_{02} &= -2\,fm'\left(A m_2 + B n_2\right) , \\ F_{03} &= -2\,fm'\left(A m_3 + B n_3\right) . \end{aligned}\right\} \tag{134}$$

EXPRESSIONS FOR THE COEFFICIENTS A AND B IN TERMS OF INVARIANTS OF Θ

In this chapter we present Halphen's theory concerning coefficients A and B with some modifications which speed up convergence of the series. Our final goal will be to express the coefficients A and B (as given by Equations 88 and 89) in terms of invariants k_1, k_2, k_3 of Θ. In order to do so we have to make use of some theorems from the theory of the Weierstrass elliptic functions. We have the basic relations

$$p'^2(u) = 4p^3(u) - g_2\,p(u) - g_3 , \tag{135a}$$

$$p''(u) = 6p^2(u) - \frac{1}{2} g_2 ; \tag{135b}$$

and we also make use of the equation

$$g_2^2 \frac{\partial \log \sigma(u)}{\partial g_3} + 18 g_3 \frac{\partial \log \sigma(u)}{\partial g_2} = \frac{3}{2} \frac{1}{\sigma(u)} \frac{\partial^2 \sigma(u)}{\partial u^2} + \frac{1}{8} g_2 u^2 \tag{136}$$

(Reference 8, p. 393).

From the defining equation for the ζ-function,

$$\frac{1}{\sigma(u)} \frac{\partial \sigma(u)}{\partial u} = \zeta(u) , \tag{137}$$

we deduce that

$$\frac{1}{\sigma(u)}\,\frac{\partial^2 \sigma(u)}{\partial u^2} - \frac{1}{\sigma^2(u)}\left(\frac{\partial \sigma(u)}{\partial u}\right)^2 = \zeta'(u)\,; \tag{138}$$

but

$$\zeta'(u) = -\,p(u)\,, \tag{139}$$

and Equation 138 can be written in the form:

$$\frac{1}{\sigma(u)}\,\frac{\partial^2 \sigma(u)}{\partial u^2} = \zeta^2(u) - p(u)\;. \tag{140}$$

Substituting this result into Equation 136 we obtain

$$g_2^{\,2}\,\frac{\partial \log \sigma(u)}{\partial g_3} + 18 g_3\,\frac{\partial \log \sigma(u)}{\partial g_2} = \frac{3}{2}\,\zeta^2(u) - \frac{3}{2}\,p(u) + \frac{1}{8}\,g_2\,u^2\;. \tag{141}$$

Introducing Halphen's linear differential operator D, defined by the equation

$$D = 12 g_3\,\frac{\partial}{\partial g_2} + \frac{2}{3}\,g_2^{\,2}\,\frac{\partial}{\partial g_3}\,, \tag{142}$$

we can write Equation 141 in the form:

$$D \log \sigma(u) = \zeta^2(u) - p(u) + \frac{1}{12}\,g_2\,u^2\;. \tag{143}$$

Differentiating this result twice with respect to u and taking the defining Equations 137 and 139 into account, we deduce that

$$D\,\zeta(u) = -\,2\,\zeta(u)\,p(u) - p'(u) + \frac{1}{6}\,g_2\,u \tag{144}$$

and

$$D\,p(u) = 2\,p'(u)\,\zeta(u) - 2\,p^2(u) + p''(u) - \frac{1}{6}\,g_2\;. \tag{145}$$

Taking Equation 135b into account, we can reduce the last result to

$$D\,p(u) = 2\,p'(u)\,\zeta(u) + 4\,p^2(u) - \frac{2}{3}\,g_2\;. \tag{146}$$

Differentiating Equation 146 with respect to u we obtain

$$D p'(u) = 6 p(u) p'(u) + 2 p''(u) \zeta(u). \tag{147}$$

If we have a function F of g_2, g_3 of the form

$$w = F(v, g_2, g_3) ,$$

and if $v = \phi(g_2, g_3)$ is also a function of g_2, g_3, then

$$Dw = \frac{\partial F}{\partial v} Dv + DF . \tag{148}$$

In forming DF on the right side of the last equation we consider v as a constant.

Designating, as before, the half of the real period by ω, we have

$$p'(\omega, g_2, g_3) = 0 , \quad \eta = \zeta(\omega, g_2, g_3) . \tag{149}$$

Putting $v = \omega, w = 0$, $F = p'(\omega, g_2, g_3)$ into Equation 148, we obtain

$$p''(\omega, g_2, g_3) D\omega + D p'(\omega) = 0 ;$$

and from the last equation it follows, by taking Equations 147 and 149 into account, that

$$p''(\omega, g_2, g_3) D\omega + 2\eta p''(\omega, g_2, g_3) = 0$$

or

$$D\omega = - 2\eta . \tag{150a}$$

Setting $w = \eta$, $F = \zeta(v, g_2, g_3)$, $v = \omega$ in Equation 148, we have

$$D\eta = \zeta'(\omega, g_2, g_3) D\omega + D\zeta(\omega, g_2, g_3) . \tag{150b}$$

From Equation 144 it follows, by taking Equation 149 into account, that

$$D\zeta(\omega, g_2, g_3) = - 2\eta p(\omega, g_2, g_3) + \frac{1}{6} g_2 \omega .$$

Substituting this into Equation 150b and taking Equations 139 and 150a into consideration we obtain

$$D\eta = + 2\eta p(\omega, g_2, g_3) - 2\eta p(\omega, g_2, g_3) + \frac{1}{6} g_2 \omega ;$$

or, finally

$$D\eta = +\frac{1}{6} g_2 \omega . \tag{151}$$

Writing the formula

$$p\left(\frac{u}{\sqrt{\mu}}, \mu^2 g_2, \mu^3 g_3\right) = \mu p(u, g_2, g_3)$$

in the form

$$p(\nu u, \nu^{-4} g_2, \nu^{-6} g_3) = \nu^{-2} p(u, g_2, g_3),$$

we conclude that if the argument u is considered to be of the first dimension, then g_2 is of the minus fourth dimension and g_3 is of the minus sixth dimension. The discriminant $\Delta = g_2^3 - 27 g_3^2$ is of the minus twelfth dimension and the absolute invariant $J = g_2^3/\Delta$ is of zero dimension. The real semi-period ω, being an argument, can be considered of the first dimension and, consequently, $x = \omega\Delta^{1/12}$ is of zero dimension. The function $p(u, g_2, g_3)$ is of the minus second dimension and then $\zeta(u, g_2, g_3)$, because of the equation $\zeta'(u) = -p(u)$, must be of the minus first dimension; in particular, $\eta = \zeta(u)$ is also of the minus first dimension and, consequently, $y' = \eta\Delta^{-1/12}$ is of zero dimension. As a consequence x and y' can be treated as functions of only the absolute invariant J. By applying the operator D to Δ and J, we deduce that

$$\begin{aligned} D\Delta &= 0, \\ DJ &= \frac{36 g_2^2 g_3}{\Delta}. \end{aligned} \tag{152}$$

Eliminating g_2 and g_3 by means of the equations $J = g_2^3/\Delta$ and $J - 1 = 27 g_3^2/\Delta$, from Equation 152, we obtain

$$DJ = 4\sqrt{3}\,\Delta^{1/6} J^{2/3} (J-1)^{1/2} . \tag{153}$$

We have, taking Equations 152 and 153 into account:

$$Dx = \frac{dx}{dJ} DJ = 4\sqrt{3}\,\frac{dx}{dJ}\Delta^{1/6} J^{2/3} (J-1)^{1/2} ; \tag{154}$$

$$Dx = \Delta^{1/12} D\omega . \tag{155}$$

By substituting $D\omega = -2\eta = -2\Delta^{1/12} y'$ (Equation 150a and definition of y') into Equation 155 and comparing the result thus obtained with the result given by Equation 154, we have

$$\frac{dx}{dJ} = -\frac{1}{2\sqrt{3}} (J-1)^{-1/2} J^{-2/3} y' . \tag{156}$$

Taking Equation 151 into account we deduce that

$$Dy' = D\left(\eta\Delta^{-1/12}\right) = \Delta^{-1/12} D(\eta) = \frac{1}{6} g_2 \omega\Delta^{-1/12} . \tag{157}$$

Similarly to Equation 154, we have for y'

$$Dy' = \frac{dy'}{dJ} DJ = 4\sqrt{3} \frac{dy'}{dJ} \Delta^{1/6} J^{2/3} (J-1)^{1/2} . \tag{158}$$

Comparing the two values of Dy', as given by Equations 157 and 158, we obtain

$$\frac{dy'}{dJ} = \frac{1}{24\sqrt{3}} \Delta^{-1/4} g_2 \omega J^{-2/3} (J-1)^{-1/2} . \tag{159}$$

Eliminating g_2 and ω from the last equation in favor of J, Δ and x by means of the relations $g_2 = \Delta^{1/3} J^{1/3}$, $\omega = x\Delta^{-1/12}$, we obtain

$$\frac{dy'}{dJ} = \frac{1}{24\sqrt{3}} (J-1)^{-1/2} J^{-1/3} x . \tag{160}$$

If we put

$$\frac{1}{2\sqrt{3}} (J-1)^{-1/2} J^{-2/3} y' = y , \tag{161}$$

then Equation 156 takes the form

$$\frac{dx}{dJ} = -y . \tag{162}$$

Eliminating y' from Equation 160 by means of the defining Equation 161, we deduce that

$$144 J (J-1) \frac{dy}{dJ} + 24 y (7J-4) - x = 0 . \tag{163}$$

Eliminating y by means of Equation 162, we now have

$$J(1-J)\frac{d^2x}{dJ^2}+\left(\frac{2}{3}-\frac{7}{6}J\right)\frac{dx}{dJ}-\frac{1}{144}x = 0 . \tag{164}$$

The last equation has the form of the hypergeometric equation

$$x(1-x)\,y'' + \left[\gamma-(\alpha+\beta+1)\,x\right]y' - \alpha\beta y = 0$$

with $\alpha = \beta = 1/12$ and $\gamma = 2/3$. Differentiating Equation 163 and again taking Equation 162 into account, we obtain a hypergeometric equation for y:

$$J(1-J)\frac{d^2y}{dJ^2}+\left(\frac{5}{3}-\frac{19}{6}J\right)\frac{dy}{dJ}-\frac{169}{144}y = 0 \tag{165}$$

with $\alpha = \beta = 13/12$ and $\gamma = 5/3$.

For the purpose of determining the coefficients A and B (as given by Equations 88 and 89) it will be convenient to use

$$X = \omega g_2^{1/4} \tag{166}$$

and

$$Y = \eta g_2^{-1/4} \tag{167}$$

rather than x and y. The next problem will be to determine hypergeometric equations of which X and Y are solutions.

Eliminating ω, η, g_2 from Equations 166 and 167 in favor of x, y, J by using

$$x = \omega\Delta^{1/12}, \quad y' = \eta\Delta^{-1/12}, \quad g_2 = J^{1/3}\Delta^{1/3}$$

and Equation 161, we obtain

$$x = J^{-1/12}X, \tag{168}$$

$$y = \frac{1}{2\sqrt{3}}(J-1)^{-1/2}J^{-7/12}Y . \tag{169}$$

Also, we introduce

$$\xi = \frac{J - 1}{J} \tag{170}$$

as a new independent variable instead of J.

The transformation of Equation 164 by the introduction of X instead of x and of ξ instead of J is a transformation of the form $y = x^{-\alpha} z$, $u = (x - 1)/x$ applied to the hypergeometric equation

$$x(1-x)\, y'' + \left[\gamma - (\alpha + \beta + 1)\, x\right] y' - \alpha\beta y = 0 .$$

We have $x = 1/(1-u)$; and by substituting

$$y = (1-u)^{\alpha}\, z ,$$

$$\frac{dy}{dx} = (1-u)^{\alpha+2} \frac{dz}{du} - \alpha (1-u)^{\alpha+1}\, z ,$$

$$\frac{d^2 y}{dx^2} = (1-u)^{\alpha+4} \frac{d^2 z}{du^2} - (2\alpha + 2)\, (1-u)^{\alpha+3} \frac{dz}{du} + \left(\alpha^2 + \alpha\right)(1-u)^{\alpha+2}\, z ,$$

into the original equation, we deduce a hypergeometric equation

$$u(1-u)\frac{d^2 z}{du^2} + \left[\gamma_1 - \left(\alpha_1 + \beta_1 + 1\right) u\right] \frac{dz}{du} - \alpha_1 \beta_1 z = 0$$

with

$$\alpha_1 = \alpha ,$$

$$\beta_1 = \alpha - \gamma + 1 ,$$

$$\gamma_1 = \alpha + \beta - \gamma + 1 .$$

In our case $\alpha = \beta = 1/12$, $\gamma = 2/3$, and $\alpha_1 = 1/12$, $\beta_1 = 5/12$, $\gamma_1 = 1/2$, and the hypergeometric equation of which X is an integral is

$$\xi(1-\xi)\frac{d^2 X}{d\xi^2} + \left(\frac{1}{2} - \frac{3}{2}\xi\right)\frac{dX}{d\xi} - \frac{5}{144} X = 0 . \tag{171}$$

A hypergeometric equation

$$x(1-x)\,y'' + \left[\gamma - (\alpha+\beta+1)\,x\right] y' - \alpha\beta y = 0$$

has the integrals

$$y_1 = F(\alpha,\ \beta,\ \gamma,\ x)\ , \tag{172}$$

$$y_2 = x^{1-\gamma}\,F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x)\ , \tag{173}$$

$$y_3 = F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x)\ , \tag{174}$$

$$y_4 = (1-x)^{\gamma-\alpha-\beta}\,F(\gamma-\beta,\ \gamma-\alpha,\ \gamma-\alpha-\beta+1,\ 1-x)\ . \tag{175}$$

If $\gamma = \alpha+\beta$, then the two last integrals coincide and an integral of the form

$$y_5 = F(\alpha,\ \beta,\ 1,\ 1-x)\ \log(1-x) + Z(1-x) \tag{176}$$

will appear instead of the integral given by Equation 175. The function $Z(1-x)$ is developable into a Taylor series in powers of $1-x$.

For Equation 171 the general integral can be written in the form

$$X = MF\left(\frac{1}{12},\ \frac{5}{12},\ \frac{1}{2},\ \xi\right) + N\xi^{1/2}\,F\left(\frac{7}{12},\ \frac{11}{12},\ \frac{3}{2},\ \xi\right), \tag{177}$$

or in the form

$$X = C_1\,F\left(\frac{1}{12},\ \frac{5}{12},\ 1,\ 1-\xi\right) + C_2\left[F\left(\frac{1}{12},\ \frac{5}{12},\ 1,\ 1-\xi\right)\log(1-\xi) + Z(1\cdot\xi)\right], \tag{178}$$

where $\xi = 1$ is a critical logarithmic point.

However, for $\xi = 1$ we have, from Equation 170, $J = \infty$ and, consequently, $\Delta = 0$. Thus, the case $\xi = 1$ corresponds to the case of a double root of the equation $4x^3 - g_2x - g_3 = 0$; and these roots become

$$e_1 = +\sqrt{\frac{g_2}{3}} \tag{179}$$

$$e_2 = e_3 = -\frac{1}{2}\sqrt{\frac{g_2}{3}}\ . \tag{180}$$

Substituting these roots into the formula for the real semi-period

$$\omega = \frac{1}{\sqrt{e_1 - e_3}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} , \qquad k^2 = \frac{e_2 - e_3}{e_1 - e_3} ,$$

we obtain

$$\omega = \frac{\pi}{\sqrt[4]{12 g_2}}$$

or

$$X = \omega \sqrt[4]{g_2} = \frac{\pi}{\sqrt[4]{12}} \quad \text{for } \xi = 1 . \tag{181}$$

Thus the value $\xi = 1$ cannot be a logarithmic singularity of X and, consequently, the value of C_2 in Equation 178 must be zero. The integral of Equation 171, which is of interest to us can be written either in the form given by Equation 177 or in the form

$$X = C_1 F\left(\frac{1}{12}, \frac{5}{12}, 1, 1-\xi\right) . \tag{182}$$

Putting $\xi = 1$ and taking Equation 181 into account, we obtain

$$\omega \sqrt[4]{g_2} = \frac{\pi}{\sqrt[4]{12}} F\left(\frac{1}{12}, \frac{5}{12}, 1, 1-\xi\right) . \tag{183}$$

A linear relation must exist between the two expressions for X (as given by Equation 177 and 183), which will help us to determine the values of the constants M and N. We can use the following relation between three solutions of the hypergeometric equation:

$$F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x) = \frac{\Gamma(\alpha+\beta-\gamma+1)\, \Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)\, \Gamma(\beta-\gamma+1)} F(\alpha, \beta, \gamma, x)$$

$$+ \frac{\Gamma(\alpha+\beta+1-\gamma)\, \Gamma(\gamma-1)}{\Gamma(\alpha)\, \Gamma(\beta)} x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x) , \tag{184}$$

which is the form given by Erdély et al. (Reference 9). By inserting into

$$\Gamma(z)\, \Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

first, $z = \gamma$ and then $z = \gamma - 1$, we obtain

$$\Gamma(\gamma - 1) = -\frac{\Gamma(\gamma)\,\Gamma(1-\gamma)}{\Gamma(2-\gamma)} .$$

Now the relation of Equation 184 can be written:

$$F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ 1-x) = \frac{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)} F(\alpha,\ \beta,\ \gamma,\ x)$$

$$-\frac{\Gamma(\alpha+\beta+1-\gamma)\,\Gamma(\gamma)\,\Gamma(1-\gamma)}{\Gamma(2-\gamma)\,\Gamma(\alpha)\,\Gamma(\beta)} x^{1-\gamma} F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x) . \qquad (185)$$

Setting $\alpha = 1/12$, $\beta = 5/12$, $\gamma = 1/2$, $x = \xi$ in Equation 185, we obtain

$$F\left(\frac{1}{12},\ \frac{5}{12},\ 1,\ 1-\xi\right) = \frac{\Gamma(1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)} F\left(\frac{1}{12},\ \frac{5}{12},\ \frac{1}{2},\ \xi\right)$$

$$-\sqrt{\xi}\,\frac{\Gamma^2\left(\frac{1}{2}\right)\,\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)\,\Gamma\left(\frac{1}{12}\right)\,\Gamma\left(\frac{5}{12}\right)} F\left(\frac{7}{12},\ \frac{11}{12},\ \frac{3}{2},\ \xi\right) . \qquad (186)$$

Putting first $n = 3$, $z = -1/12$, and then $n = 3$, $z = -1/4$ into the Gaussian formula

$$\frac{(2\pi)^{(n-1)/2}}{\sqrt{n}} = \frac{n^{nz}\,\Gamma(z+1)\,\Gamma\left(z+1-\frac{1}{n}\right)\cdots\Gamma\left(z+1-\frac{n-1}{n}\right)}{\Gamma(nz+1)} ,$$

we obtain

$$\Gamma\left(\frac{11}{12}\right)\,\Gamma\left(\frac{7}{12}\right) = \frac{2\pi}{\sqrt[4]{3}}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)},$$

$$\Gamma\left(\frac{5}{12}\right)\,\Gamma\left(\frac{1}{12}\right) = 2\pi\,\sqrt[4]{3}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} .$$

We also have

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} .$$

Substituting these values of gamma-function combinations into Equation 186, we obtain

$$F\left(\frac{1}{12}, \frac{5}{12}, 1, 1-\xi\right) = \frac{\sqrt[4]{3}}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \xi\right) - \frac{1}{\sqrt{\pi}\sqrt[4]{3}} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \xi\right)\sqrt{\xi} \cdot \tag{187}$$

Multiplying both sides of the last equation by $\pi/\sqrt[4]{12}$ and taking Equation 183 into account we have

$$X = \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \xi\right) - \frac{\sqrt{\pi}}{\sqrt{6}} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \sqrt{\xi}\, F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \xi\right) . \tag{188}$$

In order to simplify the last equation we shall make use of the B-function:

$$B(u, v) = 2\int_0^1 x^{2u-1}\left(1-x^2\right)^{v-1} dx , \tag{189}$$

and

$$B(u, v) = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)} . \tag{190}$$

Setting, in the last equations, first

$$u = \frac{1}{4}, \quad v = \frac{1}{2}$$

and then

$$u = \frac{3}{4}, \quad v = \frac{1}{2},$$

we have

$$B\left(\frac{1}{4}, \frac{1}{2}\right) = 2\int_0^1 x^{-1/2}\left(1-x^2\right)^{-1/2} dx = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$B\left(\frac{3}{4}, \frac{1}{2}\right) = 2\int_0^1 x^{1/2}\left(1-x^2\right)^{-1/2} dx = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} .$$

Taking into account

$$\frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{5}{4}\right)$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and introducing a new variable $y = \sqrt{x}$ instead of x, we have

$$\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} ,$$

$$\sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} .$$

And Equation 188 takes the form

$$X\sqrt{2} = 2A_1 F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \xi\right) - B_1\sqrt{\frac{\xi}{3}}\,F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \xi\right), \tag{191}$$

where

$$A_1 = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 1.311028777146 \cdots ,$$

$$B_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = 0.599070117367 \cdots$$

Halphen used $\Psi(\xi) = X\sqrt{2}$ instead of X. From Equation 191 we have

$$\Psi(\xi) = 2A_1 F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \xi\right) - B_1\sqrt{\frac{\xi}{3}}\,F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \xi\right), \tag{192}$$

and from Equation 183

$$\Psi(\xi) = C_1 F\left(\frac{1}{12}, \frac{5}{12}, 1, 1-\xi\right), \tag{193}$$

where $C_1 = \pi/\sqrt[4]{3}$. Goriachev used both forms of $\Psi(\xi)$.

Equation 192 is convenient if $\xi \le 1/2$ and Equation 193 is preferable if $\xi \ge 1/2$. However, remembering that a hypergeometric series converges rather slowly, we shall find it more convenient to transform Equation 193 to a fast convergent form and to use the transformed series throughout the interval $0 \le \xi \le 1$.

By applying the Goursat transformation (Reference 6)

$$F\left(\alpha, \beta, \alpha+\beta+\frac{1}{2}, z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, \alpha-\beta+\frac{1}{2}, \alpha+\beta+\frac{1}{2}, -\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

to Equation 193, we obtain

$$\Psi(\xi) = \frac{\pi}{\sqrt[4]{3}} \sqrt[6]{\frac{2}{1+\sqrt{\xi}}}\, F\left(\frac{1}{6}, \frac{1}{6}, 1, -\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\right). \tag{194}$$

The hypergeometric series which appears in the last equation is an alternating series and its convergence can be sped up considerably by applying the Euler summability process.

The general formula of the Euler summability process as applied to the series $a_0 + a_1 + a_2 + a_3 + \cdots$ can be written in the form (Reference 10):

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{N} a_k + \lim_{m\to\infty}\left[\sum_{j=1}^{m} a_{N+j} \sum_{p=j}^{m} \frac{1}{2^{p+1}} \binom{p}{j}\right]. \tag{195}$$

For the hypergeometric series

$$F(\alpha, \beta, \gamma, x) = \sum_{k=0}^{\infty} \frac{(\alpha, k)\,(\beta, k)}{(1, k)\,(\gamma, k)} x^k, \qquad (m, k) = m(m+1)\cdots(m+k-1),$$

the formula (Equation 195) takes the form:

$$F(\alpha, \beta, \gamma, x) = \sum_{k=0}^{N} \frac{(\alpha, k)\,(\beta, k)}{(1, k)\,(\gamma, k)} x^k + \lim_{m\to\infty}\left[\sum_{j=1}^{m} \frac{(\alpha, N+j)\,(\beta, N+j)}{(1, N+j)\,(\gamma, N+j)} x^{N+j} \sum_{p=j}^{m} \frac{1}{2^{p+1}} \binom{p}{j}\right].$$

Here, we have $\alpha = 1/6$, $\beta = 1/6$ and $\gamma = 1$. A high degree of approximation is already obtained by setting $N = 3$ and $m = 19$. The final result is:

$$\omega \sqrt[4]{4g_2} = \Psi(\xi) = \left(\frac{2}{1+\sqrt{\xi}}\right)^{1/6} \times$$

$$\begin{aligned} \times \big(& +2.3870942 \\ & -0.0663082\, w \\ & +0.0225632\, w^2 \\ & -0.0117691\, w^3 \\ & +0.0073743\, w^4 \\ & -0.0051060\, w^5 \\ & +0.0037250\, w^6 \\ & -0.0027325\, w^7 \\ & +0.0019070\, w^8 \\ & -0.0011936\, w^9 \\ & +0.0006337\, w^{10} \\ & -0.0002710\, w^{11} \\ & +0.0000884\, w^{12} \\ & -0.0000205\, w^{13} \\ & +0.0000030\, w^{14} \\ & -0.0000002\, w^{15} \big) , \end{aligned} \tag{196}$$

where

$$w = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}} ,$$

and the convergence is fast. The program written by A. J. Smith, Jr. of the Theoretical Division, Goddard Space Flight Center makes use of Equations 196 and 207.*

A comparison of Goriachev's computations for the minor planet (1) Ceres (done on the basis of Equations 192 and 193) with the results obtained by Smith using Equation 196 shows a complete agreement between the two methods of computation. The coefficient B (Equation 89)

$$B = \frac{2\omega}{\pi g_2} ,$$

*This program will be discussed in the following Part 2 of this treatise, by A. J. Smith, Jr.

which appears in the expression for the dyadic Φ (Equation 113) and in the expressions for the disturbing force F_0 (Equations 114b and 134), can now be put into the final form. We have, taking Equation 196 into account,

$$B = \frac{\sqrt{2}}{\pi g_2 \sqrt[4]{g_2}} \Psi(\xi) . \tag{197}$$

We have still to form a convenient expression for the coefficient A (Equation 88)

$$A = \frac{16}{\pi\Delta}\left(\frac{3}{2}\frac{g_3}{g_2}\omega - \eta\right) .$$

We have

$$\xi = \frac{27\, g_3^2}{g_2^3} , \tag{198}$$

$$D\xi = D\frac{27\, g_3^2}{g_2^3} = D\left(1 - \frac{1}{J}\right) = \frac{1}{J^2} DJ \quad ; \tag{199}$$

and by considering Equation 153 we deduce

$$D\xi = 4 \sqrt{3}\, \Delta^{1/6} J^{-4/3} (J-1)^{1/2} \tag{200}$$

from the last equation. Eliminating J and $J-1$ in favor of g_2 and g_3 by means of $J = g_2^3/\Delta$ and $J - 1 = 27\, g_3^2/\Delta$, we deduce from Equation 200 that

$$D\xi = \frac{36\, g_3}{g_2^4} \Delta . \tag{201}$$

We also have

$$D\left(g_2^{1/4}\right) = \frac{3\, g_3}{g_2} \sqrt[4]{g_2} \tag{202}$$

and $D\omega = -2\eta$ (Equation 150a).

Let us now apply the operator D to $\omega \sqrt[4]{4g_2} = \Psi(\xi)$. Taking Equations 201, 202, and 150a into account, we deduce that

$$-2\eta \sqrt[4]{4g_2} + \omega \frac{3\, g_3}{g_2} \sqrt[4]{4g_2} = \frac{36\, g_3}{g_2^4} \cdot \Delta \cdot \Psi'(\xi)$$

or

$$\frac{16}{\Delta}\left(\frac{3}{2}\ \frac{g_3}{g_2}\ \omega - \eta\right) = \frac{144\sqrt{2}}{\sqrt[4]{g_2}}\ \frac{1}{g_2^{\ 4}}\ g_3\ \Psi'\ (\xi)\ ,$$

and thus

$$A = \frac{144\ g_3\sqrt{2}}{\pi\ g_2^{\ 4}\ \sqrt[4]{g_2}}\ \Psi'\ (\xi)\ . \tag{203}$$

Now by applying the formula

$$\frac{d}{dx}\ F(\alpha,\ \beta,\ \gamma,\ x) = \frac{\alpha\beta}{\gamma}\ F(\alpha+1,\ \beta+1,\ \gamma+1,\ x)$$

to $\Psi(\xi)$ as given by Equation 193 we obtain a value of $\Psi'\ (\xi)$ which may be substituted into Equation 203 to yield

$$A = -\frac{5g_3\sqrt{2}}{g_2^{\ 4}\ \sqrt[4]{3g_2}}\ F\left(\frac{13}{12}\ ,\ \frac{17}{12}\ ,\ 2,\ 1-\xi\right)\ . \tag{204}$$

Taking the equation

$$\xi = \frac{J-1}{J} = \frac{27\ g_3^{\ 2}}{g_2^{\ 3}}$$

nto account, we deduce from Equation 204, by eliminating g_3 in favor of ξ and g_2, that

$$A = -\frac{\sqrt[4]{12}}{9}\ .\ \frac{\sqrt[4]{g_2}}{g_2^{\ 3}}\ \sqrt{\xi}\ F\left(\frac{13}{12}\ ,\ \frac{17}{12}\ ,\ 2,\ 1-\xi\right)\ . \tag{205}$$

By applying the Goursat transformation

$$F\left(\alpha,\ \beta,\ \alpha+\beta-\frac{1}{2}\ ,\ z\right)$$

$$= (1-z)^{-1/2}\ \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2\alpha}\ F\left(2\alpha-1,\ \alpha-\beta+\frac{1}{2}\ ,\ \alpha+\beta-\frac{1}{2}\ ,\ -\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

to the hypergeometric series in the right side of Equation 205, we obtain, finally,

$$A = -\frac{5}{\sqrt[4]{3}}\left(\frac{2}{1+\sqrt{\xi}}\right)^{7/6} F\left(\frac{1}{6}, \frac{7}{6}, 2, -w\right)\frac{\sqrt{6}}{9}\frac{\sqrt[4]{g_2}}{g_2^3}. \qquad (206)$$

The Euler summability process can be applied to the hypergeometric series of the last equation. We have

$$\begin{aligned} -\frac{5}{\sqrt[4]{3}} F\left(\frac{1}{6}, \frac{7}{6}, 2, -w\right) = & -3.7991784 \\ & + 0.3693646 \; w \\ & - 0.1556119 \; w^2 \\ & + 0.0889726 \; w^3 \\ & - 0.0586828 \; w^4 \\ & + 0.0419870 \; w^5 \\ & - 0.0313364 \; w^6 \\ & + 0.0233758 \; w^7 \\ & - 0.0165247 \; w^8 \\ & + 0.0104483 \; w^9 \\ & - 0.0055933 \; w^{10} \\ & + 0.0024083 \; w^{11} \\ & - 0.0007898 \; w^{12} \\ & + 0.0001837 \; w^{13} \\ & - 0.0000268 \; w^{14} \\ & + 0.0000018 \; w^{15} , \end{aligned} \qquad (207)$$

where, as before, $w = (1-\sqrt{\xi})/(1+\sqrt{\xi})$. Equation 207 was used instead of Goriachev's tables in the actual computations.

EQUATIONS FOR SECULAR VARIATIONS OF ELEMENTS

Let $fm'S$, $fm'T$, $fm'Z$ be the radial, the tangential, and the normal components respectively of the disturbing force F. We have (Reference 11) for the variation of elliptic elements $a, e, \pi = \omega + \Omega$, $L = g + \pi$, i, and Ω

$$\frac{da}{dt} = \frac{2m' na^2}{M+m}\left(S\frac{ae}{\sqrt{1-e^2}}\sin v + T\frac{a^2}{r}\sqrt{1-e^2}\right);$$

$$\frac{de}{dt} = \frac{m'}{M+m}\cdot\frac{na\left(1-e^2\right)}{e}\left(S\frac{ae}{\sqrt{1-e^2}}\sin v + T\frac{a^2}{r}\sqrt{1-e^2}\right) - \frac{m'}{M+m}\frac{na\sqrt{1-e^2}}{e} Tr ;$$

$$\frac{d\pi}{dt} = \frac{m'}{M+m} \cdot \frac{na\sqrt{1-e^2}}{e}\left[- S a \cos v + Ta \left(1 + \frac{1}{1-e^2} \cdot \frac{r}{a}\right) \sin v\right] + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt} ;$$

$$\frac{dL}{dt} = - \frac{m'}{M+m} 2 nar S + \left(1 - \sqrt{1-e^2}\right)\frac{d\pi}{dt} + 2\sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt} ;$$

$$\frac{di}{dt} = \frac{m'}{M+m} \cdot \frac{na}{\sqrt{1-e^2}} Zr \cos (v + \omega) ;$$

$$\sin i \frac{d\Omega}{dt} = \frac{m'}{M+m} \cdot \frac{na}{\sqrt{1-e^2}} Zr \sin (v + \omega) .$$

Let S_0, T_0, Z_0 be the values of S, T, Z averaged over the orbit of the disturbing body:

$$S_0 = \frac{1}{2\pi}\int_0^{2\pi} S \, dg' ,$$

$$T_0 = \frac{1}{2\pi}\int_0^{2\pi} T \, dg' ,$$

$$Z_0 = \frac{1}{2\pi}\int_0^{2\pi} Z \, dg' .$$

Averaging the equations for the variation of elements with respect to g and g' and taking the equations

$$r \cos v = a \cos E - ae ,$$

$$r \sin v = a\sqrt{1-e^2} \sin E ,$$

$$r = a - ae \cos E ,$$

$$E - e \sin E = g ,$$

$$dg = \frac{r}{a} dE$$

into account, we deduce the following equations for secular variations of elliptic elements:

$$\frac{da}{dt} = \frac{2m' na^3}{M+m} \cdot \frac{1}{2\pi}\int_0^{2\pi} \left(e S_0 \sin E + T_0 \sqrt{1-e^2}\right) dE ;$$

$$\frac{de}{dt} = \frac{m' na^2\sqrt{1-e^2}}{M+m} \cdot \frac{1}{2\pi}\int_0^{2\pi} \left[S_0 \sqrt{1-e^2} \sin E + T_0\left(-\frac{3}{2} e + 2\cos E - \frac{1}{2} e \cos 2E\right)\right] dE ;$$

$$\frac{d\pi}{dt} = \frac{m' na^2 \sqrt{1-e^2}}{(M+m)\, e} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[- S_0 (\cos E - e) \right.$$

$$\left. + T_0 \left(1 + \frac{1}{1-e^2} \cdot \frac{r}{a}\right) \sqrt{1-e^2} \sin E \right] dE + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt} ;$$

$$\frac{dL}{dt} = - \frac{2m' na^2}{M+m} \cdot \frac{1}{2\pi} \int_0^{2\pi} S_0 \frac{r^2}{a^2} dE + \frac{e}{1+\sqrt{1-e^2}} \left(e \frac{d\pi}{dt}\right) + 2\sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt} ;$$

$$\frac{di}{dt} = \frac{m' na^2}{(M+m)\sqrt{1-e^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} Z_0 \frac{r}{a} \left[(\cos E - e) \cos\omega - \sqrt{1-e^2} \sin E \sin\omega\right] dE ;$$

$$\sin i \frac{d\Omega}{dt} = \frac{m' na^2}{(M+m)\sqrt{1-e^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} Z_0 \frac{r}{a} \left[(\cos E - e) \sin\omega + \sqrt{1-e^2} \sin E \cos\omega\right] dE .$$

The values of S_0, T_0, Z_0 are computed analytically by using the formulas developed in the previous two sections. The integrals with respect to E are computed numerically by giving E a set of particular values conveniently distributed over the orbit of the disturbed body.

The secular variation of da/dt is zero in this theory and in the process of computation the smallness of da/dt will determine the range of validity of the theory and, at the same time, serve as a check of the accuracy of the computation.

CONCLUSION

The collection of formulas given in Appendix A was programmed for the actual computation of long range effects in the motion of artificial satellites and minor planets using step by step integration. Halphen's method can be especially useful when near-resonance conditions arise. In this case difficulties are caused by the presence of a small divisor of the form $i\dot{\omega} + i'\dot{\omega}' + j\dot{\Omega} + j'\dot{\Omega}'$ if the problem is treated analytically.

In the case of an artificial satellite Halphen's method might give information on the long range effects and the stability of orbit over the interval of approximately 15-20 years. In the case of minor planets, it can supply information about the long range ("secular") effects in the elements over intervals of hundreds of thousands of years; the integration step can be taken to be 100-500 years. We assume that no sharp commensurability between mean motions of the disturbed and disturbing bodies exists. The secular variations of the elements of the disturbing bodies are also taken into consideration.

ACKNOWLEDGMENT

The author would like to take the opportunity to express his gratitude to Mr. Arthur J. Smith, Jr. who programmed the theory and without whose generous assistance the numerical part of the work could never have been completed.

REFERENCES

1. Brown, E. W., and Shook, C. A., "Planetary Theory," Cambridge (England): The University Press, 1933, p. 159.

2. Gauss, K. F., "Determinatio attractionis quam in punctum quodvis positionis datae exerceret planeta si eius massa per totam orbitam ratione temporis quo singulae partes describuntur uniformiter esset dispertita," (1818) in: His *Werke*, Göttingen: Königliche Gesellschaft der Wissenschaften, 1866, Vol. 3, pp. 331-335.

3. Halphen, G. H., "Traité des Fonctions Elliptiques et de Leurs Applications," Vol. 2, Paris: Gauthier-Villars et Fils, 1888.

4. Musen, P., "On the Long-Period Lunisolar Effect in the Motion of the Artificial Satellite," *J. Geophys. Res.* 66(6):1659-1665, June 1961.

5. Goriachev, N. N., "Halphen's Method for the Computation of Secular Perturbations," University of Tomsk, 1937 (In Russian).

6. Goursat, É., "Sur l'Équation Différentielle Linéaire qui Admet pour Intégrale la Série Hypergéométrique," *Annales Scientifiques de l'École Normale Supérieure, Supplément* Ser. 2, 10:3-142, 1881.

7. Gibbs, J. W., "Vector Analysis," New Haven: Yale University Press, 1901.

8. Hancock, H., "Lectures on the Theory of Elliptic Functions: Analysis," New York: Dover, 1958.

9. Bateman Manuscript Project, California Institute of Technology, "Higher Transcendental Functions," Vol. 1, New York: McGraw-Hill, 1953.

10. Knopp, K., "Infinite Sequences and Series," New York: Dover, 1956, p. 144.

11. Brown, E. W., "An Introductory Treatise on the Lunar Theory," Cambridge (England): The University Press, 1896, p. 63.

Appendix A

COLLECTION OF FORMULAS

I. Elements of the disturbed planet:

$$g_0,\ a,\ n,\ i,\ \omega,\ \Omega,\ e,\ b.$$

Elements of the disturbing planet:

$$g_0',\ a',\ n',\ i',\ \omega'\ \ \Omega',\ e',\ b',\ .$$

II.

$$\begin{bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{bmatrix} = \begin{bmatrix} +\cos\Omega & -\sin\Omega & 0 \\ +\sin\Omega & +\cos\Omega & 0 \\ 0 & 0 & +1 \end{bmatrix} \cdot \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos i & -\sin i \\ 0 & +\sin i & +\cos i \end{bmatrix} \cdot \begin{bmatrix} +\cos\omega & -\sin\omega & 0 \\ +\sin\omega & +\cos\omega & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$$\begin{bmatrix} P_x' & Q_x' & R_x' \\ P_y' & Q_y' & R_y' \\ P_z' & Q_z' & R_z' \end{bmatrix} = \begin{bmatrix} +\cos\Omega' & -\sin\Omega' & 0 \\ +\sin\Omega' & +\cos\Omega' & 0 \\ 0 & 0 & +1 \end{bmatrix} \cdot \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos i' & -\sin i' \\ 0 & +\sin i' & +\cos i' \end{bmatrix} \cdot \begin{bmatrix} +\cos\omega' & -\sin\omega' & 0 \\ +\sin\omega' & +\cos\omega' & 0 \\ 0 & 0 & +1 \end{bmatrix} .$$

III. For example, for $E = 0°, 10°, 20°, \cdots, 350°$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x' & P_y' & P_z' \\ Q_x' & Q_y' & Q_z' \\ R_x' & R_y' & R_z' \end{bmatrix} \cdot \begin{bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{bmatrix} \cdot \begin{bmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\,\sin E \\ 0 \end{bmatrix};$$

$$\alpha = x + e'a', \quad \beta = y, \quad \gamma = z.$$

IV.

$$k_1 = \alpha^2 + \beta^2 + \gamma^2 - \left(a'^2 + b'^2\right)$$

$$k_2 = a'^2 b'^2 - b'^2 \alpha^2 - a'^2 \beta^2 - \left(a'^2 + b'^2\right)\gamma^2$$

$$k_3 = a'^2 b'^2 \gamma^2$$

$$g_2 = \frac{4}{3}\left(k_1^2 - 3k_2\right)$$

$$g_3 = \frac{4}{27}\left(2k_1^3 - 9k_1 k_2 + 27k_3\right)$$

$$\xi = \frac{27 g_3^2}{g_2^3}$$

$$k_4 = 9k_3 - k_1 k_2 ,$$

$$k_5 = k_1\left(k_1 k_2 - 3k_3\right) - 2k_2^2 .$$

V.

$$A = \frac{\sqrt{6}\,\sqrt[4]{g_2}}{9 g_2^3} \cdot \frac{144}{\pi}\sqrt{\xi}\,\Psi'(\xi)$$

$$B = \frac{\sqrt{2}}{\pi g_2 \sqrt[4]{g_2}}\,\Psi(\xi)$$

$$w = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}$$

$$\Psi(\xi) = \left(\frac{2}{1+\sqrt{\xi}}\right)^{1/6} \times \Big(+2.3870942 - 0.0663082\,w + 0.0225632\,w^2 - 0.0117691\,w^3 + 0.0073743\,w^4 - 0.0051060\,w^5 + 0.0037250\,w^6 - 0.0027325\,w^7 + 0.0019070\,w^8 - 0.0011936\,w^9 + 0.0006337\,w^{10} - 0.0002710\,w^{11} + 0.0000884\,w^{12} - 0.0000205\,w^{13} + 0.0000030\,w^{14} - 0.0000002\,w^{15}\Big)$$

$$\frac{144}{\pi}\sqrt{\xi}\,\Psi'(\xi) = \left(\frac{2}{1+\sqrt{\xi}}\right)^{7/6} \times \Big(-3.7991784 + 0.3693646\,w - 0.1556119\,w^2 + 0.0889726\,w^3 - 0.0586828\,w^4 + 0.0419870\,w^5 - 0.0313364\,w^6 + 0.0233758\,w^7 - 0.0165247\,w^8 + 0.0104483\,w^9 - 0.0055933\,w^{10} + 0.0024083\,w^{11} - 0.0007898\,w^{12} + 0.0001837\,w^{13} - 0.0000268\,w^{14} + 0.0000018\,w^{15}\Big)$$

VI.

$$a_{11} = k_4 \left(\alpha^2 - a'^2\right) + k_5 + \frac{3}{2}\frac{g_2 k_3}{a'^2} ,$$

$$a_{22} = k_4 \left(\beta^2 - b'^2\right) + k_5 + \frac{3}{2}\frac{g_2 k_3}{b'^2} ,$$

$$a_{33} = k_4 \gamma^2 + k_5 + \frac{3}{2} g_2 \left(\alpha^2 b'^2 + \beta^2 a'^2 - a'^2 b'^2\right) ,$$

$$a_{12} = a_{21} = k_4 \alpha\beta ,$$

$$a_{23} = a_{32} = k_4 \beta\gamma - \frac{3}{2} g_2 a'^2 \beta\gamma ,$$

$$a_{31} = a_{13} = k_4 \gamma\alpha - \frac{3}{2} g_2 b'^2 \gamma\alpha ,$$

$$a'_{11} = \alpha^2 - a'^2 - \frac{1}{3} k_1 ,$$

$$a'_{22} = \beta^2 - b'^2 - \frac{1}{3} k_1 ,$$

$$a'_{33} = \gamma^2 - \frac{1}{3} k_1 ,$$

$$a'_{12} = a'_{21} = \alpha\beta ,$$

$$a'_{23} = a'_{32} = \beta\gamma ,$$

$$a'_{31} = a'_{13} = \gamma\alpha ,$$

$$A_{ij} = a_{ij} A + a'_{ij} B \qquad (i, j = 1, 2, 3)$$

$$A_{ij} = A_{ji} .$$

VII.

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = -2 \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

VIII.

$$r = a(1 - e \cos E)$$

$$\alpha_1 = \frac{x}{r},$$

$$\beta_1 = \frac{y}{r},$$

$$\gamma_1 = \frac{z}{r},$$

$$\begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} P_x' & P_y' & P_z' \\ Q_x' & Q_y' & Q_z' \\ R_x' & R_y' & R_z' \end{bmatrix} \cdot \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$$

$$\alpha_2 = \gamma_1 \beta_3 - \beta_1 \gamma_3$$

$$\beta_2 = \alpha_1 \gamma_3 - \gamma_1 \alpha_3$$

$$\gamma_2 = \beta_1 \alpha_3 - \alpha_1 \beta_3 .$$

IX.

$$\begin{bmatrix} S_0 \\ T_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \cdot \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} .$$

X.

$$\frac{da}{dt} = \frac{2m' na^3}{M+m} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left(e S_0 \sin E + T_0 \sqrt{1-e^2} \right) dE ,$$

$$\frac{de}{dt} = \frac{m' na^2 \sqrt{1-e^2}}{M+m} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[S_0 \sqrt{1-e^2} \sin E + T_0 \left(-\frac{3}{2} e + 2\cos E - \frac{1}{2} e \cos 2E \right) \right] dE ,$$

$$\frac{d\pi}{dt} = \frac{m' na^2 \sqrt{1-e^2}}{(M+m)\, e} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[-S_0 (\cos E - e) + T_0 \left(1 + \frac{1}{1-e^2} \cdot \frac{r}{a} \right) \sqrt{1-e^2} \sin E \right] dE + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt} ,$$

$$\frac{dL}{dt} = -\frac{2m' na^2}{M+m} \cdot \frac{1}{2\pi} \int_0^{2\pi} S_0 \cdot \frac{r^2}{a^2} dE + \frac{e}{1+\sqrt{1-e^2}} \left(e \frac{d\pi}{dt} \right) + 2\sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt} ,$$

$$\frac{di}{dt} = \frac{m' na^2}{(M+m)\sqrt{1-e^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} Z_0 \frac{r}{a} \left[(\cos E - e) \cos\omega - \sqrt{1-e^2} \sin E \sin\omega \right] dE ,$$

$$\sin i \frac{d\Omega}{dt} = \frac{m' na^2}{(M+m)\sqrt{1-e^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} Z_0 \frac{r}{a} \left[(\cos E - e) \sin\omega + \sqrt{1-e^2} \sin E \cos\omega \right] dE .$$

NASA-Langley, 1963 G-319